Curry-Howard Isomorphisms without Conversions

Author: Please provide author information

Abstract

We present three term calculi that respectively embody classical, intuitionistic and linear logics. These term calculi are equipped with reductions that satisfy subject reduction, confluence and strong normalisation. An advantage of our term calculi is their *simplicity*: They dispense with the tangled (commuting) conversions in existing term calculi, where conversions are one of the fundamental problems in proof theory. This simplicity also means the *abstraction* of our approach that discards the inessential syntactic details. By formulating the computations for the three logics in this abstract level, the present work deepens the computational analyses of the logics.

2012 ACM Subject Classification Theory of computation → Type theory

Keywords and phrases Curry-Howard isomorphisms, type theory, term calculi, linear logic

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

1 Introduction

▶ **Notation 1.** *Let* $\mathbb{N}$ *be the set of all* numerals*, and let* $\mathcal{T}$ *be the set of all* (term) variables*. We use capital letters* A*,* B*, etc. for formulas, capital Greek letters* Δ*,* Γ*, etc. for sequences of formulas, lower case letters* x*,* y*, etc. for variables, and vectors* $\vec{x}$*,* $\vec{y}$*, etc. for sequences of variables. We abbreviate* $x_1 : A_1, \ldots, x_n : A_n$ *as* $\vec{x} : \Gamma$ *if* $\vec{x} = x_1, \ldots, x_n$ *and* $\Gamma = A_1, \ldots, A_n$*.*

1.1 Background and Motivation

1.1.1 The Bureaucracy of Syntax

Gentzen's *sequent calculi* [8] exhibit a beautiful symmetry in logic and offer a powerful method to study the meta-theoretic properties of logic. However, they contain *inessential details* on the representation of proofs. For instance, the two different derivation trees

$$(\textsc{Cut})\;\frac{\overset{u}{\Delta \vdash A, \Xi} \qquad (\vee\mathrm{R})\;\dfrac{\overset{v}{\Gamma, A \vdash B, \Theta}}{\Gamma, A \vdash B \vee C, \Theta}}{\Delta, \Gamma \vdash B \vee C, \Xi, \Theta} \qquad\qquad (\vee\mathrm{R})\;\frac{(\textsc{Cut})\;\dfrac{\overset{u}{\Delta \vdash A, \Xi} \qquad \overset{v}{\Gamma, A \vdash B, \Theta}}{\Delta, \Gamma \vdash B, \Xi, \Theta}}{\Delta, \Gamma \vdash B \vee C, \Xi, \Theta}$$

in the sequent calculus LK [8] for classical logic should represent *the same proof* since in both of the trees the rule Cut is applied to A, and the right rule $\vee$R on disjunction $\vee$ to B and C. Girard [10] calls this fundamental problem *the bureaucracy of syntax.*

This problem also complicates the computation or *cut-elimination* for sequent calculi. For example, a standard cut-elimination algorithm for LK transforms the left tree to the right one by permuting the order of the two rules. As already indicated, however, this permutation does not change the *essence* of the proof. Besides, due to this business of permuting rules, it is in general nontrivial to verify the correctness of cut-elimination algorithms.

Another classic work by Gentzen is *natural deductions* [8]. They are, unlike sequent calculi, asymmetric yet closer to the actual reasonings by mathematicians. Although natural deductions are more satisfactory for a representation of proofs than sequent calculi, they are also bound by the bureaucracy of syntax if they subsume falsity ff or disjunction $\vee$ [11, §10].

> First problem: The bureaucracy of syntax prohibits an ideal representation of proofs and complicates computations.

42nd Conference on Very Important Topics (CVIT 2016).
Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:20
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1.1.2 The Curry-Howard Isomorphisms

The *Curry-Howard isomorphisms* [16] refer to a variety of correspondences between formal systems in mathematical logic and models of computation in theoretical computer science. The correspondences enable fruitful exchanges of ideas and results between the two fields as well as their unification. For instance, having their origin in Church's *(simply-typed) λ-calculus* [6], a variety of *term calculi* or *type theories* today can be seen not only as functional programming languages but also as formal systems via the Curry-Howard isomorphisms.

However, even though the λ-calculus is an almost perfect computational counterpart of the natural deduction NJ [8] for intuitionistic logic, it is bound by inessential details on terms as soon as it subsumes falsity ff or disjunction $\vee$ as in the case of NJ [11, §10]. Let us illustrate this problem by the following terms of the λ-calculus:

$$(\vee\mathrm{E})\,\dfrac{(\mathrm{Ax})\,\dfrac{}{z : A \vee B \vdash z : A \vee B} \qquad (\vee\mathrm{I})\,\dfrac{(\mathrm{Ax})\,\dfrac{}{x : A, y : B \vdash x : A}}{x : A, y : B \vdash \mathrm{inj}_1^D(x) : A \vee B}}{z : A \vee B \vdash \mathrm{let}\ z\ \mathrm{be}\ [x, y]\ \mathrm{in}\ \mathrm{inj}_1^D(z) : A \vee B}$$

$$(\vee\mathrm{I})\,\dfrac{(\vee\mathrm{E})\,\dfrac{(\mathrm{Ax})\,\dfrac{}{z : A \vee B \vdash z : A \vee B} \qquad (\mathrm{Ax})\,\dfrac{}{x : A, y : B \vdash x : A}}{z : A \vee B \vdash \mathrm{let}\ z\ \mathrm{be}\ [x, y]\ \mathrm{in}\ x : A}}{z : A \vee B \vdash \mathrm{inj}_1^D(\mathrm{let}\ z\ \mathrm{be}\ [x, y]\ \mathrm{in}\ x) : A \vee B}$$

These terms should be identified because only the difference is the order of applying rules, i.e., the terms differ only *inessentially* in the places of the constructs for the rules $\vee$I and $\vee$E.

To address this problem, the λ-calculus is equipped with *(commuting) conversions* between these inessentially different terms [11, §10.6]. However, the conversions are rather tangled. More fundamentally, this method does not solve the problem in any intrinsic sense because it merely takes the equivalence classes of terms modulo the inessential details [11, p. 73].

Because classical logic is more powerful than intuitionistic logic and more widely used by mathematicians, Curry-Howard isomorphisms for classical logic is of great interest (for the theoretical computer scientists who aim to design disciplined yet powerful models of computation as well as for the mathematical logicians who aim to constructivise a wider range of mathematics). However, this is a highly nontrivial task, and existing term calculi for classical logic are much more complex than the λ-calculus. For instance, Parigot's *$\lambda\mu$-calculus* [14], which is the λ-calculus equipped with constructs on *μ-variables*, corresponds to a variant of a natural deduction with multiple conclusions [13] for classical logic, where μ-variables are introduced to handle the multiple conclusions. However, the reductions on μ-variables are quite involved. Also, the $\lambda\mu$-calculus necessitates very intricate conversions for μ-variables [5, §2]. Another line of work is the *$\overline{\lambda}\mu\tilde{\mu}$-calculus* introduced by Curien and Herbeline [7] for the implication fragment of LK, later extended to entire LK by Wadler [19]. This work dispenses with the conversions of the $\lambda(\mu)$-calculus, but instead it needs *ς-reduction* [19, Figures 7–8], another conversion, to facilitate β-reduction in such a way that it satisfies confluence.

Finally, linear logic refines classical and intuitionistic logics in the sense that their logical constants and connectives are decomposed into more primitive ones in linear logic [9]. Thus, it is expected that term calculi for linear logic similarly refine term calculi for classical and intuitionistic logics, so they are of great interest too. However, the fine structure of linear logic poses challenges in achieving term calculi for linear logic; in fact, it is nontrivial to design term calculi for linear logic that satisfy even basic properties such as *closure under substitution* and *confluence*. For instance, the pioneering term calculus for intuitionistic linear logic by Abramsky [1] is not even closed under substitution. Benton et al. [3] solved this problem by another term calculus for intuitionistic linear logic. Barber and Plotkin [2] pointed out, however, that this term calculus by Benton et al. is *ad-hoc* since its rules on exponential ! do not follow the standard format of natural deductions, and they proposed yet another, *dual context* term calculus whose rules on exponential follow the standard format. However, their calculus is not a definite solution either as its reduction is not even confluent [2, §7.2]. Besides, these calculi necessitate more conversions than the λ-calculus. The *linear* $\lambda\mu$*-calculus* suffers from conversions even more complex than those on the $\lambda\mu$-calculus [5, §3].

> Second problem: Term calculi for classical and linear logics suffer from the plethora of complex constructs/conversions or the lack of basic computational properties.

1.1.3 A Unity of Term Calculi

As listed above, there have been a variety of term calculi in the literature, and there has been no single formalism of term calculi applicable *uniformly* to classical, intuitionistic and linear logics. For instance, we have seen that it is not straightforward to apply the formalism of the λ-calculus to classical or linear logic, e.g., one needs to add μ-variables, dual contexts or new conversions. In contrast, sequent calculi are readily applicable to the three logics.

This *schism* of term calculi between classical, intuitionistic and linear logics makes it hard to compare and relate the computations underlying the three logics. In contrast, the uniform format of sequent calculi makes the interrelations between the logics quite transparent. We would like to extend such a uniform format of the logics to their *computations*.

> Third problem: The lack of a uniform format of term calculi hampers a deeper analysis on the interrelations between the computations underlying classical, intuitionistic and linear logics, or a practical synthesis of the computations.

1.2 Main Results

The aim of the present work is to establish a solution to the three fundamental problems listed in the previous section. We focus on *simple types* and *propositions* as the first step; we leave it as future work to extend the present work to *dependent types* and *predicates*.

Specifically, our solution consists of three term calculi such that

1. They share the same format and achieve Curry-Howard isomorphisms respectively for classical, intuitionistic and (classical) linear logics;
2. They are simple and abstract, e.g., free from μ-variables, conversions or dual contexts;
3. They are closed under substitution, and their reductions enjoy subject reduction, confluence and strong normalisation.

1.3 Key Ideas

We list key ideas underlying our solution in the next few subsections progressively so that the rest of the present article is more or less just an implementation of the ideas.

1.3.1 Two-Sided Term Calculi

We introduce three term calculi respectively for classical, intuitionistic and linear logics, called the *classical* σ*-calculus*, the *intuitionistic* σ*-calculus* and the *linear* σ*-calculus*, respectively.

The Greek letter σ signifies the symmetric or *two-sided* nature of these term calculi.

We design our term calculi in such a way that they correspond to sequent calculi because the format of sequent calculi is applicable *uniformly* to classical, intuitionistic and linear logics. Note that the term calculi by Curien et al. [7, 19] also correspond to sequent calculi, but *not completely* since their terms are not symmetric: Only variables may occur on the left-hand side of their terms. One of our key ideas is then to allow non-variables to occur on the left-hand side, leading to our two-sided term calculi. Concretely, we abstract derivation trees for a sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ in sequent calculi into two-sided terms

$$p_1 : A_1, \ldots, p_n : A_n \dashv \mathrm{cut}(c_1 \mid r_1), \ldots, \mathrm{cut}(c_k \mid r_k) \vdash b_1 : B_1, \ldots, b_m : B_m, \tag{1}$$

where the left-hand side $p_1 : A_1, \ldots, p_n : A_n$ corresponds to left rules in sequent calculi, the middle part $\mathrm{cut}(c_1 \mid r_1), \ldots, \mathrm{cut}(c_k \mid r_k)$ to cuts, and the right-hand side $b_1 : B_1, \ldots, b_m : B_m$ to right rules. We abbreviate the terms (1) as $\delta : \Delta \dashv \Pi \vdash \phi : \Phi$, where $\delta = p_1, \ldots, p_n$, $\Delta = A_1, \ldots, A_n$, $\Pi = \mathrm{cut}(c_1 \mid r_1), \ldots, \mathrm{cut}(c_k \mid r_k)$, $\phi = b_1, \ldots, b_m$ and $\Phi = B_1, \ldots, B_m$. These terms are two-sided in the sense that the construct δ may contain non-variables. Our three term calculi all share this basic format. Note that derivation trees are only *auxiliary* in our term calculi, and terms, not derivation trees, are what represents proofs.

The two-sided terms ignore the inessential order of applying rules, solving the bureaucracy of sequent calculi. For instance, the trees of the example given in §1.1.1 are translated into

$$(\textsc{Cut})\ \dfrac{\begin{matrix} u' \\ \delta : \Delta \dashv \Pi \vdash a : A, \phi : \Phi \end{matrix} \qquad (\vee\mathrm{R})\ \dfrac{\begin{matrix} v' \\ \gamma : \Gamma, p : A \dashv \Sigma \vdash b : B, \psi : \Psi \end{matrix}}{\gamma : \Gamma, p : A \dashv \Sigma \vdash \iota_1(b) : B \vee C, \psi : \Psi}}{\delta : \Delta, \gamma : \Gamma \dashv \Pi, \mathrm{cut}(a \mid p), \Sigma \vdash \iota_1(b) : B \vee C, \psi : \Psi}$$

$$(\vee\mathrm{R})\ \dfrac{(\textsc{Cut})\ \dfrac{\begin{matrix} u' \\ \delta : \Delta \dashv \Pi \vdash a : A, \phi : \Phi \end{matrix} \qquad \begin{matrix} v' \\ \gamma : \Gamma, p : A \dashv \Sigma \vdash b : B, \psi : \Psi \end{matrix}}{\delta : \Delta, \gamma : \Gamma \dashv \Pi, \mathrm{cut}(a \mid p), \Sigma \vdash b : B, \psi : \Psi}}{\delta : \Delta, \gamma : \Gamma \dashv \Pi, \mathrm{cut}(a \mid p), \Sigma \vdash \iota_1(b) : B \vee C, \psi : \Psi}$$

in the classical σ-calculus, where u' and v' correspond to u and v, respectively, ι_1 is the constructor for the right rule $\vee$R on disjunction, and cut for the rule Cut. Crucially, they derive *the same term*. In this way, our terms achieve an abstract representation of proofs by focusing on *how formulas are inhabited* while ignoring the order of applying rules.

Note also that we have nothing to compute between these terms, unlike the example of sequent calculi given in §1.1.1, because they are already the same.

1.3.2 Superposition of Side Constructs

Nevertheless, our two-sided term calculi bring a nontrivial problem on (additive) conjunction $\wedge$ and disjunction $\vee$. For instance, while the introduction rule

$$(\wedge\mathrm{I})\ \dfrac{\vec{x} : \Delta \vdash a : A \qquad \vec{x} : \Delta \vdash b : B}{\vec{x} : \Delta \vdash \langle a, b \rangle : A \wedge B}$$

on conjunction of the λ-calculus is simple thanks to the sharing of the context $\vec{x} : \Delta$, this principle is unavailable for our term calculi as non-variables may occur on the left-hand side.

We solve this problem by an idea based on *slicing* invented by Girard [9]. The idea is quite simple: We regard the introduction rule $\wedge$I on conjunction as the operation of *superposing* the terms in the hypotheses, and similarly for the introduction rule on disjunction. Concretely, the corresponding right-rule $\wedge$R of the classical σ-calculus is, approximately,

$$(\wedge\mathrm{R})\ \frac{\delta : \Delta \dashv \Pi \vdash a : A, \phi : \Phi \qquad \delta' : \Delta \dashv \Pi' \vdash b : B, \phi' : \Phi}{\{\delta[i], \delta'[j]\} : \Delta \dashv \{\Pi[i], \Pi'[j]\} \vdash \{a\wedge_i, \wedge_j b\} : A \wedge B, \{\phi[i], \phi'[j]\}}$$

where the numerals $i, j \in \mathbb{N}$ are the *identifiers* of the application of $\wedge$R. In this way, we take the superposition of the two terms in the hypotheses by attaching the constructors $(\_)\wedge_i$ and $\wedge_j(\_)$ as well as the *tags* $[i]$ and $[j]$. The tags $[i]$ and $[j]$ are to keep track of the side constructs superposed in this way, which is necessary for implementing our reduction (§1.3.4). Accordingly, just like the *variable convention* in the λ-calculus, we identify our terms modulo the choice of the numerals, and we always assume that they are chosen to be *fresh*. The dual rule on disjunction of the classical σ-calculus is formulated in the same fashion.

In order to implement this idea systematically, we generalise the basic format (1) to

$$\mathscr{P}_1 : A_1, \dots, \mathscr{P}_n : A_n \dashv \{\sigma_1[\mathcal{S}_1], \dots, \sigma_k[\mathcal{S}_k]\} \vdash \mathscr{B}_1 : B_1, \dots, \mathscr{B}_m : B_m, \qquad (2)$$

where $\mathscr{P}_l$ $(1 \leqslant l \leqslant n)$ and $\mathscr{B}_r$ $(1 \leqslant r \leqslant m)$ are finite multisets of p's and b's in (1), respectively, equipped with finite sets of numerals, called *synchronisations*, and $\{\sigma_1[\mathcal{S}_1], \dots, \sigma_k[\mathcal{S}_k]\}$ is a finite multiset of cuts $\sigma_h := \mathrm{cut}(\mathscr{C}_h \mid \mathscr{R}_h)$ $(1 \leqslant h \leqslant k)$ in (1) equipped with synchronisations $\mathcal{S}_h$. Abusing notation, we again write $\delta : \Delta \dashv \Pi \vdash \phi : \Phi$ for the new format (2).

To be precise, a *multiset* is a pair (S, m) of a set S and a map m from S to the set of all positive integers. The idea is that S is the underlying set of the multiset, and $m(e)$ is the number of occurrences of each $e \in S$ in the multiset. Given another multiset (T, o), the *disjoint union* $(S, m) \uplus (T, o)$ is defined by $(S, m) \uplus (T, o) := (S \cup T, m + o)$, where $(m + o)(e) := m(e) + o(e)$ for all $e \in S \cup T$. A *submultiset* of (S, m) is a multiset (S', m') that satisfies $S' \subseteq S$ and $m'(e') \leqslant m(e')$ for all $e' \in S'$. For brevity, we represent a multiset by listing its elements $\{e_1, \dots, e_n\}$ *with multiplicity* and write $\emptyset$ for the empty (multi)set.

1.3.3 Explicit Structural Rules without Conversions

The generalised format (2) also enables us to dispense with constructors for structural rules in a quite pleasing way. For example, we define the left weakening rule WL and the left contraction rule CL of the classical σ-calculus *explicitly* and *abstractly* by

$$(\mathrm{WL})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}{\delta : \Delta, \emptyset : A \dashv \Pi \vdash \phi : \Phi} \qquad (\mathrm{CL})\ \frac{\delta : \Delta, \mathscr{P} : A, \mathscr{P}' : A \vdash \phi : \Phi}{\delta : \Delta, \mathscr{P} \uplus \mathscr{P}' : A \vdash \phi : \Phi}$$

and similarly for the right structural rules. That is, we inhabit weakening and contraction by the empty multiset $\mathrm{wk} := \emptyset$ and the disjoint union $\mathrm{ct}(\mathscr{P}, \mathscr{P}') := \mathscr{P} \uplus \mathscr{P}'$, respectively.

One pleasing feature of this formulation is that it satisfies the desired equations

$$\mathrm{ct}(\mathscr{P}, \mathscr{P}') = \mathrm{ct}(\mathscr{P}', \mathscr{P}) \qquad \mathrm{ct}(\mathscr{P}, \mathrm{wk}) = \mathscr{P} \qquad \mathrm{ct}(\mathscr{P}, \mathrm{ct}(\mathscr{P}', \mathscr{P}'')) = \mathrm{ct}(\mathrm{ct}(\mathscr{P}, \mathscr{P}'), \mathscr{P}'')$$

without relying on conversions. Note that Selinger's *control categories* [15], one of the most established categorical semantics of classical logic, require these equations though we leave it as future work to study the categorical counterpart of our term calculi. This categorical semantics indicates that the equations are desirable from the semantic standpoint.

▶ Remark 2. We write $\top$ and 1 for the units of tensor $\otimes$ and with &, respectively, i.e., we swap the original notations [9] similarly to Troelstra [18, §2.7], as we find it more systematic.

Another pleasing feature of this implementation of structural rules is that, by defining the rules for additive units 1 and 0 of the linear σ-calculus to be

$$(1\mathrm{R})\ \frac{}{\vec{\emptyset} : \Delta \dashv \emptyset \vdash o : 1, \vec{\emptyset} : \Phi} \qquad (0\mathrm{L})\ \frac{}{\vec{\emptyset} : \Delta, \zeta : 0 \dashv \emptyset \vdash \vec{\emptyset} : \Phi}$$

where o and ζ are the constructors for the respective rules, the derivation trees

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\dashv \emptyset \vdash o : 1}\,(1\mathrm{R})}{\emptyset : !A \dashv \emptyset \vdash o : 1}\,(\mathrm{WL})}{\emptyset : !A, \emptyset : !A \dashv \emptyset \vdash o : 1}\,(\mathrm{WL})}{\emptyset : !A \dashv \emptyset \vdash o : 1}\,(\mathrm{CL}) \qquad \dfrac{\dfrac{\dfrac{}{\zeta : 0 \dashv \emptyset \vdash \emptyset : ?A, \emptyset : ?A}\,(0\mathrm{L})}{\zeta : 0 \dashv \emptyset \vdash \emptyset : ?A}\,(\mathrm{CL})}{\zeta : 0 \dashv \emptyset \vdash \emptyset : ?A, \emptyset : ?A}\,(\mathrm{WL})$$

of the linear σ-calculus generate *the same terms* as those given by the singleton applications of the rules 1R and 0L, respectively, *without relying on conversions*. This is again desirable from the categorical standpoint since the units 1 and 0 should be terminal and initial, respectively.

1.3.4 Reductions and Lafont's Critical Pairs

Corresponding to cut-elimination, our reduction proceeds by eliminating cuts in the middle of the format (2). For instance, we equip the classical σ-calculus with the reduction

$$\delta[i] \uplus \delta'[j] \dashv \Pi[i] \uplus \Pi'[j] \uplus \{\mathrm{cut}(\mathscr{A} \wedge_i \mid \pi_1(\mathscr{P}))\} \vdash \phi[i] \uplus \phi'[j] \to \delta \dashv \Pi \uplus \{\mathrm{cut}(\mathscr{A} \mid \mathscr{P})\} \vdash \phi$$

$$\delta[i] \uplus \delta'[j] \dashv \Pi[i] \uplus \Pi'[j] \uplus \{\mathrm{cut}(\wedge_j \mathscr{B} \mid \pi_2(\mathscr{Q}))\} \vdash \phi[i] \uplus \phi'[j] \to \delta' \dashv \Pi' \uplus \{\mathrm{cut}(\mathscr{B} \mid \mathscr{Q})\} \vdash \phi'$$

where we omit formulas. Note that the tags $[i]$ and $[j]$ identify the constructs to be eliminated by the reduction. In the following, we write $\heartsuit\{\clubsuit/\spadesuit\}$ for the *substitution* of $\clubsuit$ for $\spadesuit$ in $\heartsuit$.

There are, however, subtle issues on the reductions whose cut formulas are inhabited by structural rules. For instance, how should we reduce the following cuts?

$$\delta : \Delta \dashv \Pi \uplus \{\mathrm{cut}(\emptyset \mid \mathscr{P})\} \vdash \phi : \Phi \to \ldots? \qquad \delta : \Delta \dashv \Pi \uplus \{\mathrm{cut}(\mathscr{A} \mid \emptyset)\} \vdash \phi : \Phi \to \ldots?$$

We cannot use standard cut-elimination as it is on derivation trees, while we work on terms.

Our solution is to attach an identifier $(\_)_i$ to the main construct of *each* rule, i.e., not only additives, and the tag $[i]$ to the side constructs. As a result, we can reduce the cuts by

$$\delta : \Delta \dashv \Pi \uplus \{\mathrm{cut}(\emptyset \mid \mathscr{P})\} \vdash \phi : \Phi \to (\delta : \Delta \dashv \Pi \vdash \phi)\{\emptyset/f(\mathscr{P})\} : \Phi$$

$$\delta : \Delta \dashv \Pi \uplus \{\mathrm{cut}(\mathscr{A} \mid \emptyset)\} \vdash \phi : \Phi \to (\delta : \Delta \dashv \Pi \vdash \phi)\{\emptyset/f(\mathscr{A})\} : \Phi,$$

where $f(\mathscr{P})$ is any construct with a tag $[i]$ such that the identifier $(\_)_i$ occurs in $\mathscr{P}$, and similarly for $f(\mathscr{A})$. (Strictly speaking, the substitutions $\{\emptyset/f(\mathscr{P})\}$ and $\{\emptyset/f(\mathscr{A})\}$ must be *recursive*; see Definition 12.) In other words, the tags keep track of the constructs to be deleted. A similar method works for cut formulas inhabited by contraction as we shall see.

Even if we somehow attach an identifier $(\_)_i$ to the empty multiset $\emptyset$, however, we have, similarly to Lafont's *critical pair* [11, pp. 150–151], a *dilemma* in reducing the cuts

$$\delta : \Delta \dashv \Pi \uplus \{\mathrm{cut}(\emptyset \mid \emptyset)\} \vdash \phi : \Phi \to \ldots?$$

because the two cases just described conflict with each other, leading to different outputs. We have a similar problem on cuts whose cut formulas are both inhabited by contraction.

Our solution for this problem is to restrict the inhabitant $\mathscr{A}$ of the cut formula A on the left-hand side to a *singleton* multiset $\mathscr{A} = \{a_i[\mathcal{S}]\}$ so that structural rules are possible only on the cut formulas on the right-hand side. Of course, the other choice is also possible, i.e., the same restriction on the inhabitant $\mathscr{P}$ of the cut formula A on the right-hand side.

Strictly speaking, this restriction is also applied to the superpositions given by additive conjunction and disjunction, but it is undesirable (e.g., this restriction bans the rule Cut whose left hypothesis is given by the left rule $\vee$L). To address this problem, we generalise the concept of singleton multisets in such a way that they are preserved by additive conjunction and disjunction, and impose the restriction *with respect to these generalised singleton multisets* (via *quantities* in Definition 5). In this way, the restriction is applied only to structural rules.

1.4 Our Contributions and Related Work

The most important contribution is the novel *format* (2) of our two-sided term calculi. This format is *uniformly* applicable to classical, intuitionistic and linear logics, and achieves the Curry-Howard isomorphisms on the sequent calculi for them in a straightforward fashion. Besides, this format realises an *abstract* representation of proofs in the sense that they are free from conversions. This abstraction is highly nontrivial because even the λ-calculus is bound by conversions, and existing term calculi for classical and linear logics necessitate even more conversions (§1.1.2). Girard et al. [11, §10] emphasise that conversions are one of the fundamental problems in proof theory. Our two-sided format solves this problem.

Another contribution is the *reductions*. By dispensing with the business of permuting rules, our reductions are much simpler than cut-elimination. As a result, it is quite straightforward to show their basic properties such as subject reduction, confluence and strong normalisation. In addition, our reductions satisfy *diamond property*, which does not hold in the λ-calculus.

As related work, we have listed existing term calculi for classical, intuitionistic and linear logics in the literature in §1.1. In addition, the term calculi for linear logic by Wadler [20] and Bierman [4] are also *two-sided*, so they are more closely related to the present work. The unpublished work by Wadler [20] also employs multisets or *stacks* of constructs as inhabitants of formulas, but his structural rules, rule Cut and reduction are quite different from ours. Besides, additives and exponentials are missing in Bierman [4].

Finally, *proof nets* [9] also achieve an abstract representation of proofs and cut-elimination for linear logic, and they have been extended to additives (yet not units or exponentials) [12]. As pointed out by Bierman [5, pp. 44–45], however, proof nets impose the De Morgan dualities *on the nose*, which is not desirable from the programming standpoint.

1.5 The Structure of the Present Article

We present our term calculus together with a reduction for linear logic in §2, and for classical and intuitionistic logics in §3. Finally, we draw a conclusion and propose future work in §4.

2 A Term Calculus for Linear Logic

Having described the main ideas in §1.3, we can now assemble these ideas into term calculi. We first introduce a term calculus for (classical) linear logic [9], called the *linear σ-calculus*, together with a reduction, called the *$\ell\beta$-reduction*. It is a reformulation of the (two-sided) sequent calculus for linear logic [18], where the $\ell\beta$-reduction corresponds to cut-elimination.

▶ Remark 3. Following Troelstra [18] and Bierman [5], we do not impose the definitional *De Morgan equalities* of linear logic because, as pointed out by Bierman [5, pp. 44–45], these *strict* equalities between formulas are not desirable from the programming standpoint.

▶ **Definition 4** (synchronisations). Synchronisations $\mathcal{S}$, $\mathcal{T}$, *etc. are finite subsets of* $\mathbb{N}$.

▶ **Definition 5** (the linear σ-calculus). *The* linear σ-calculus *consists of the following:*

- (Types) Types A, B, *etc. are formulas of (classical) linear logic [9], i.e., the expressions defined by the grammar*

$$A, B := X \mid \top \mid \bot \mid 1 \mid 0 \mid A \otimes B \mid A \parr B \mid A \,\&\, B \mid A \oplus B \mid A^\perp \mid !A \mid ?A$$

where X ranges over the set $\mathcal{P}$ of propositional variables, we define $A \multimap B := A^\perp \parr B$, *and we call* $\top$ top, $\bot$ bottom, 1 one, 0 zero, $\otimes$ tensor, $\parr$ par, $\&$ with, $\oplus$ plus, $(\_)^\perp$ linear negation, $!$ of-course, $?$ why-not, *and* $\multimap$ linear implication.

- ((Co)patterns) Patterns $\mathscr{P}, \mathscr{Q}$, *etc. and* copatterns $\mathscr{A}, \mathscr{B}$, *etc. are the finite multisets defined by the grammar*

$$p_i, q_i := \underline{x}_i \mid \tau_i \mid \beta_i \mid \zeta_i \mid \mathscr{P} \otimes_i \mathscr{Q} \mid \mathscr{P} \parr_i \mathscr{Q} \mid \pi^{\&}_{j,i}(\mathscr{P}) \mid \mathscr{P} \oplus_i \mid \oplus_i \mathscr{Q} \mid \mathscr{A}^{\perp_i} \mid !_i \mathscr{P} \mid ?_i \mathscr{P}$$
$$a_i, b_i := \overline{x}_i \mid \tau_i \mid \beta_i \mid o_i \mid \mathscr{A} \otimes_i \mathscr{B} \mid \mathscr{A} \parr_i \mathscr{B} \mid \mathscr{A} \&_i \mid \&_i \mathscr{B} \mid \iota^{\oplus}_{j,i}(\mathscr{A}) \mid \mathscr{P}^{\perp_i} \mid !_i \mathscr{A} \mid ?_i \mathscr{A}$$
$$\mathscr{P}, \mathscr{Q} := \emptyset \mid \mathscr{P} \uplus \mathscr{Q} \mid \{p_i[\emptyset]\} \mid \mathscr{P}[+i]$$
$$\mathscr{A}, \mathscr{B} := \emptyset \mid \mathscr{A} \uplus \mathscr{B} \mid \{a_i[\emptyset]\} \mid \mathscr{A}[+i]$$

where x ranges over the set $\mathcal{T}$ of all term variables, i over $\mathbb{N}$, and j over $\{0,1\}$, and $\mathscr{P}[+i] := \{\, p_k[\mathcal{S} \cup i] \mid p_k[\mathcal{S}] \in \mathscr{P} \,\}$, where $i \notin \mathcal{S}$ by the convention below, and similarly for $\mathscr{A}[+i]$. We represent a (multi)set by listing its elements (with multiplicity). We use the letter e as a meta-variable for p and a. We call the subscripts $(\_)_i$ identifiers*, and the synchronisation elements i in the bracket $[\_]$* tags*. We usually omit the empty tag $[\emptyset]$.*
- (Quantities) *An occurrence of a numeral i in $e_k[\mathcal{S}]$ is* superposed *if it is the identifier $(\_)_i$ on $\&$ or $\oplus$, or the corresponding tag $[i]$. Let $\sharp(e_k[\mathcal{S}])$ be the number of superposed occurrences in $e_k[\mathcal{S}]$. The* quantity $\sharp\mathscr{M}$ *of a (co)pattern $\mathscr{M}$ is then defined by*

$$\sharp\mathscr{M} := \sum_{e_k[\mathcal{S}] \in \mathscr{M}} 1/2^{\sharp(e_k[\mathcal{S}])}.$$

▶ Remark 6. The point of quantities is that among the typing rules in Figure 1 below only the structural rules can generate an inhabitant $\mathscr{M}$ of a type such that $\sharp\mathscr{M} \neq 1$. This is a key to implement the restriction on cuts in a desired way as explained in §1.3.4.

- ((Co)contexts) Contexts *are finite sequences $\mathscr{P}_1 : A_1, \ldots, \mathscr{P}_n : A_n$, for which we write $\delta : \Delta$ if $\delta = \mathscr{P}_1, \ldots, \mathscr{P}_n$ and $\Delta = A_1, \ldots, A_n$, and* cocontexts *are those $\mathscr{B}_1 : B_1, \ldots, \mathscr{B}_m : B_m$, for which we write $\phi : \Phi$ if $\phi = \mathscr{B}_1, \ldots, \mathscr{B}_m$ and $\Phi = B_1, \ldots, B_m$. We extend operations on (co)patterns to (co)contexts componentwisely.*
- (Cuts and states) Cuts σ, ς, *etc. and* states Π, Σ, *etc. are defined by the grammar*

$$\sigma, \varsigma := \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \qquad\qquad \Pi, \Sigma := \emptyset \mid \Pi \uplus \Sigma \mid \{\sigma[\emptyset]\} \mid \Pi[+i]$$

where $\Pi[+i] := \{\, \sigma[\mathcal{S} \cup i] \mid \sigma[\mathcal{S}] \in \Pi \,\}$ and again $i \notin \mathcal{S}$ by the convention below.
- (Raw terms) Linear raw-terms *are expressions of the form $\delta : \Delta \dashv \Pi \vdash \phi : \Phi$.*
- (Terms) Linear σ-terms *from Δ to Φ are linear raw-terms of the form $\delta : \Delta \dashv \Pi \vdash \phi : \Phi$ derivable by the typing rules in Figure 1, where we modify numerals and term variables along the derivation, when necessary, in such a way that they are always fresh.*

▶ Remark 7. The convention on linear σ-terms is the same as the *variable convention* in the λ-calculus. Thus, we identify linear σ-terms *modulo numerals and term variables.* This convention is applied to the classical and the intuitionistic σ-calculi introduced later too.

▶ **Example 8.** Some examples of linear σ-terms are displayed in Figures 2–3, where we omit formulas in the conclusion of Figure 3 for lack of space.

Because the typing rules of the linear σ-calculus correspond precisely to the derivation rules of the sequent calculus for linear logic [18, §2.2], we have:

▶ **Proposition 9** (a CHI for linear logic). *A sequent $\Delta \vdash \Phi$ is derivable in the sequent calculus for linear logic if and only if there is a linear σ-term $\delta : \Delta \dashv \Pi \vdash \phi : \Phi$ for some δ, Π and ϕ, where the sequent is derivable without cut if and only if $\Pi = \emptyset$.*

As discussed in [3], it is a nontrivial problem to design a term calculus for linear logic that enjoys *the closure under substitution.* The linear σ-calculus passes this test:

$$(\mathrm{XL})\ \frac{\delta : \Delta, \mathscr{P} : A, \mathscr{Q} : B, \gamma : \Gamma \dashv \Pi \vdash \phi : \Phi}{\delta : \Delta, \mathscr{Q} : B, \mathscr{P} : A, \gamma : \Gamma \dashv \Pi \vdash \phi : \Phi} \qquad (\mathrm{XR})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi, \mathscr{A} : A, \mathscr{B} : B, \psi : \Psi}{\delta : \Delta \dashv \Pi \vdash \phi : \Phi, \mathscr{B} : B, \mathscr{A} : A, \psi : \Psi}$$

$$(!\mathrm{W})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}{\delta : \Delta, \emptyset : !A \dashv \Pi \vdash \phi : \Phi} \qquad (?\mathrm{W})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}{\delta : \Delta \dashv \Pi \vdash \emptyset : ?A, \phi : \Phi}$$

$$(!\mathrm{C})\ \frac{\delta : \Delta, \mathscr{P} : !A, \mathscr{P}' : !A \dashv \Pi \vdash \phi : \Phi}{\delta : \Delta, \mathscr{P} \uplus \mathscr{P}' : !A \dashv \Pi \vdash \phi : \Phi} \qquad (?\mathrm{C})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : ?A, \mathscr{A}' : ?A, \phi : \Phi}{\delta : \Delta \dashv \Pi \vdash \mathscr{A} \uplus \mathscr{A}' : ?A, \phi : \Phi}$$

$$(!\mathrm{D})\ \frac{\delta : \Delta, \mathscr{P} : A \dashv \Pi \vdash \phi : \Phi}{\delta[+i] : \Delta, !_i\mathscr{P} : !A \dashv \Pi[+i] \vdash \phi[+i] : \Phi} \qquad (?\mathrm{D})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash ?_i\mathscr{A} : ?A, \phi[+i] : \Phi}$$

$$(?\mathrm{L})\ \frac{\delta : !\Delta, \mathscr{P} : A \dashv \Pi \vdash \phi : ?\Phi}{\delta[+i] : !\Delta, ?_i\mathscr{P} : ?A \dashv \Pi[+i] \vdash \phi[+i] : ?\Phi} \qquad (!\mathrm{R})\ \frac{\delta : !\Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : ?\Phi}{\delta[+i] : !\Delta \dashv \Pi[+i] \vdash !_i\mathscr{A} : !A, \phi[+i] : ?\Phi}$$

$$(\textsc{Cut})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi \qquad \gamma : \Gamma, \mathscr{P} : A \dashv \Sigma \vdash \psi : \Psi}{\delta : \Delta, \gamma : \Gamma \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \Sigma \vdash \phi : \Phi, \psi : \Psi}\ (\sharp\mathscr{A} = 1)$$

$$(\textsc{Ax})\ \frac{}{\underline{x}_i[j] : A \dashv \emptyset \vdash \overline{x}_j[i] : A} \qquad (1\mathrm{R})\ \frac{}{\vec{\emptyset} : \Delta \dashv \emptyset \vdash o_i : 1, \vec{\emptyset} : \Phi} \qquad (0\mathrm{L})\ \frac{}{\vec{\emptyset} : \Delta, \zeta_i : 0 \dashv \emptyset \vdash \vec{\emptyset} : \Phi}$$

$$(\top\mathrm{L})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}{\delta[+i] : \Delta, \tau_i : \top \dashv \Pi[+i] \vdash \phi[+i] : \Phi} \qquad (\top\mathrm{R})\ \frac{}{\dashv \emptyset \vdash \tau_i : \top}$$

$$(\bot\mathrm{L})\ \frac{}{\beta_i : \bot \dashv \emptyset \vdash} \qquad (\bot\mathrm{R})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash \beta_i : \bot, \phi[+i] : \Phi}$$

$$(\otimes\mathrm{L})\ \frac{\delta : \Delta, \mathscr{P} : A, \mathscr{Q} : B \dashv \Pi \vdash \phi : \Phi}{\delta[+i] : \Delta, \mathscr{P} \otimes_i \mathscr{Q} : A \otimes B \dashv \Pi[+i] \vdash \phi[+i] : \Phi}$$

$$(\otimes\mathrm{R})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi \qquad \gamma : \Gamma \dashv \Sigma \vdash \mathscr{B} : B, \psi : \Psi}{\delta[+i] : \Delta, \gamma[+i] : \Gamma \dashv \Pi[+i] \uplus \Sigma[+i] \vdash \mathscr{A} \otimes_i \mathscr{B} : A \otimes B, \phi[+i] : \Phi, \psi[+i] : \Psi}$$

$$(\parr\mathrm{L})\ \frac{\delta : \Delta, \mathscr{P} : A \dashv \Pi \vdash \phi : \Phi \qquad \gamma : \Gamma, \mathscr{Q} : B \dashv \Sigma \vdash \psi : \Psi}{\delta[+i] : \Delta, \gamma[+i] : \Gamma, \mathscr{P} \parr_i \mathscr{Q} : A \parr B \dashv \Pi[+i] \uplus \Sigma[+i] \vdash \phi[+i] : \Phi, \psi[+i] : \Psi}$$

$$(\parr\mathrm{R})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \mathscr{B} : B, \phi : \Phi}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash \mathscr{A} \parr_i \mathscr{B} : A \parr B, \phi[+i] : \Phi}$$

$$(\&\mathrm{L})\ \frac{\delta : \Delta, \mathscr{P} : A_j \dashv \Pi \vdash \phi : \Phi}{\delta[+i] : \Delta, \pi^{\&}_{j,i}(\mathscr{P}) : A_1 \& A_2 \dashv \Pi[+i] \vdash \phi[+i] : \Phi}\ (j \in \{1, 2\})$$

$$(\&\mathrm{R})\ \frac{\delta_1 : \Delta \dashv \Pi_1 \vdash \mathscr{A}_1 : A_1, \phi_1 : \Phi \qquad \delta_2 : \Delta \dashv \Pi_2 \vdash \mathscr{A}_2 : A_2, \phi_2 : \Phi}{\delta_1[+i] \uplus \delta_2[+j] : \Delta \dashv \Pi_1[+i] \uplus \Pi[+j] \vdash \mathscr{A}_1\&_i, \&_j\mathscr{A}_2 : A_1 \& A_2, \phi_1[+i] \uplus \phi_2[+j] : \Phi}$$

$$(\oplus\mathrm{L})\ \frac{\delta_1 : \Delta, \mathscr{P}_1 : A_1 \dashv \Pi_1 \vdash \phi_1 : \Phi \qquad \delta_2 : \Delta, \mathscr{P}_2 : A_2 \dashv \Pi_2 \vdash \phi_2 : \Phi}{\delta_1[+i] \uplus \delta_2[+j] : \Delta, \mathscr{P}_1\oplus_i, \oplus_j\mathscr{P}_2 : A_1 \oplus A_2 \dashv \Pi_1[+i] \uplus \Pi_2[+j] \vdash \phi_1[+i] \uplus \phi_2[+j] : \Phi}$$

$$(\oplus\mathrm{R})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A_j, \phi : \Phi}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash \iota^{\oplus}_{j,i}(\mathscr{A}) : A_1 \oplus A_2, \phi[+i] : \Phi}\ (j \in \{1, 2\})$$

$$((\_)^{\perp}\mathrm{L})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi}{\delta[+i] : \Delta, \mathscr{A}^{\perp_i} : A^{\perp} \dashv \Pi[+i] \vdash \phi[+i] : \Phi}$$

$$((\_)^{\perp}\mathrm{R})\ \frac{\delta : \Delta, \mathscr{P} : A \dashv \Pi \vdash \phi : \Phi}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash \mathscr{P}^{\perp_i} : A^{\perp}, \phi[+i] : \Phi}$$

Figure 1 The typing rules of the linear σ-calculus

$$\dfrac{\dfrac{\dfrac{}{\underline{x}_1[2] : A \dashv \emptyset \vdash \overline{x}_2[1] : A}\,(\mathrm{Ax})}{\pi^{\&}_{1,5}(\underline{x}_1[2]) : A \,\&\, B \dashv \emptyset \vdash \overline{x}_2[1,5] : A}\,(\&\mathrm{L}) \quad \dfrac{\dfrac{}{\underline{y}_3[4] : B \dashv \emptyset \vdash \overline{y}_4[3] : B}\,(\mathrm{Ax})}{\pi^{\&}_{2,6}(\underline{y}_3[4]) : A \,\&\, B \dashv \emptyset \vdash \overline{y}_4[3,6] : B}\,(\&\mathrm{L})}{\dfrac{\pi^{\&}_{1,5}(\underline{x}_1[2])[7], \pi^{\&}_{2,6}(\underline{y}_3[4])[8] : A \,\&\, B \dashv \emptyset \vdash \overline{x}_2[1,5]\&_7, \&_8(\overline{y}_4[3,6]) : A \,\&\, B}{\pi^{\&}_{1,5}(\underline{x}_1[2])[7,9], \pi^{\&}_{2,6}(\underline{y}_3[4])[8,9] : A \,\&\, B \dashv \emptyset \vdash \iota^{\oplus}_{1,9}(\overline{x}_2[1,5]\&_7, \&_8(\overline{y}_4[3,6])) : (A \,\&\, B) \oplus C}\,(\oplus\mathrm{R})}\,(\&\mathrm{R})$$

Figure 2 The first example of a linear σ-term

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\underline{z}_{11}[12] : A \dashv \emptyset \vdash \overline{z}_{12}[11] : A}\,(\mathrm{Ax})}{\pi_{1,13}(\underline{z}_{11}[12]) : A \,\&\, B \dashv \emptyset \vdash \overline{z}_{12}[11,13] : A}\,(\&\mathrm{L}) \quad \dfrac{\dfrac{}{\underline{w}_{15}[16] : A \dashv \emptyset \vdash \overline{w}_{16}[15] : A}\,(\mathrm{Ax})}{\pi_{1,14}(\underline{w}_{15}[16]) : A \,\&\, B \dashv \emptyset \vdash \overline{w}_{16}[15,14] : A}\,(\&\mathrm{L})}{\pi_{1,13}(\underline{z}_{11}[12])[17], \pi_{1,14}(\underline{w}_{15}[16])[18] : A \,\&\, B \dashv \emptyset \vdash \overline{z}_{12}[11,13]\&_{17}, \&_{18}(\overline{w}_{16}[15,14]) : A \,\&\, B}\,(\&\mathrm{R}) \quad (\text{Figure 2})}{\pi_{1,13}(\underline{z}_{11}[12])[17], \pi_{1,14}(\underline{w}_{15}[16])[18] \dashv \mathrm{cut}(\overline{z}_{12}[11,13]\&_{17}, \&_{18}(\overline{w}_{16}[15,14]) \mid \pi^{\&}_{1,5}(\underline{x}_1[2])[7,9], \pi^{\&}_{2,6}(\underline{y}_3[4])[8,9]) \vdash \iota^{\oplus}_{1,9}(\overline{x}_2[1,5]\&_7, \&_8(\overline{y}_4[3,6]))}\,(\mathrm{Cut})$$

Figure 3 The second example of a linear σ-term

▶ **Lemma 10** (exponential lemma). *If $\delta : \Delta \dashv \Pi \vdash \mathscr{A} : !A, \phi : \Phi$ and $\gamma : \Gamma, \mathscr{P} : ?A \dashv \Sigma \vdash \psi : \Psi$ are linear σ-terms, then $\Delta = !\Delta'$, $\Phi = ?\Phi'$, $\Gamma = !\Gamma$ and $\Psi = ?\Psi'$ for some Δ', Φ', Γ' and Ψ'.*

Proof. By induction on the typing rules in Figure 1. ◀

▶ **Theorem 11** (linear substitution). *If $\delta : \Delta \dashv \Pi \vdash \uplus_i a_i[\mathcal{S}_i] : A, \phi : \Phi$, $\sharp(\uplus_i a_i[\mathcal{S}_i]) = 1$, and $\gamma : \Gamma, \uplus_k \underline{x_k}[\mathcal{T}_k] : A \dashv \Sigma \vdash \psi : \Psi$ are linear σ-terms, then so is $\uplus_k \delta[+\mathcal{T}_k] : \Delta, \uplus_{i,k} \gamma\{a_i/\overline{x}_k\}[+\mathcal{S}_i] : \Gamma \dashv (\uplus_k \Pi[+\mathcal{T}_k]) \uplus (\uplus_{i,k} \Sigma\{a_i/\overline{x}_k\}[+\mathcal{S}_i]) \vdash \uplus_k \phi[+\mathcal{T}_k] : \Phi, \uplus_{i,k} \psi\{a_i/\overline{x}_k\}[+\mathcal{S}_i] : \Psi$.*

Proof. By induction on the linear σ-term $\gamma : \Gamma, \uplus_k \underline{x_k}[\mathcal{T}_k] : A \dashv \Sigma \vdash \psi : \Psi$, where we apply Lemma 10 to the cases of the rules ?L and !R. ◀

We next define a β-reduction on the linear σ-calculus, called the *$\ell\beta$-reduction*, which corresponds to cut-elimination for sequent calculi. Recall that cut-elimination *deletes* or *duplicates* derivation subtrees if a cut formula is given by an additive rule, weakening or contraction. However, this method is unavailable for the present work because we are working on terms, not derivation trees. As discussed in §1.3.4, we solve this problem by identifiers and tags (Definition 5). Specifically, we implement the $\ell\beta$-reduction corresponding to the cut-elimination mentioned above by the following two operations, which respectively delete and duplicate suitable parts of a linear raw-term with the help of identifiers and tags:

▶ **Definition 12** (recursive deletion). *Let F be a linear raw-term, and $e_i[\mathcal{S}]$ be an occurrence in F. The recursive deletion of $e_i[\mathcal{S}]$ in F is the following operation:*

1. *Define the set $(F, e_i[\mathcal{S}])^{\mathcal{D}}_0 := \{e_i[\mathcal{S}]\}$;*
2. *Define $(F, e_i[\mathcal{S}])^{\mathcal{D}}_{n+1}$ to be the set of all occurrences $e''_k[\mathcal{U}]$ in F such that $j \in \mathcal{U}$ for some element $e'_j[\mathcal{T}] \in (F, e_i[\mathcal{S}])^{\mathcal{D}}_n$;*
3. *Iterate the step 2 until $(F, e_i[\mathcal{S}])^{\mathcal{D}}_{n+1} = \emptyset$, and obtain a linear raw-term $F\langle\emptyset/e_i[\mathcal{S}]\rangle$ from F by replacing each element of $(F, e_i[\mathcal{S}])^{\mathcal{D}} := \bigcup_{l=0}^{n}(F, e_i[\mathcal{S}])^{\mathcal{D}}_l$ with $\emptyset$ in F.*

▶ **Definition 13** (recursive copying). *Let F be a linear raw-term, and $e_i[\mathcal{S}]$ be an occurrence in F. The recursive copying of $e_i[\mathcal{S}]$ in F is the following operation:*

1. *Define the sets $(F, e_i[\mathcal{S}])^{\mathcal{C}}_0 := \{e_i[\mathcal{S}]\}$ and $(F, e^{\star}_{i^{\star}}[\mathcal{S}])^{\mathcal{C}}_0 := \{e^{\star}_{i^{\star}}[\mathcal{S}]\}$, where $e^{\star}_{i^{\star}}[\mathcal{S}]$ is obtained from $e_i[\mathcal{S}]$ by replacing variables and identifiers (but not tags) with fresh ones;*
2. *Define $(F, e_i[\mathcal{S}])^{\mathcal{C}}_{n+1}$ to be the set of all occurrences $e''_k[\mathcal{U}]$ in F such that $j \in \mathcal{U}$ for some element $e'_j[\mathcal{T}] \in (F, e_i[\mathcal{S}])_n$ and obtain another set $(F, e^{\star}_{i^{\star}}[\mathcal{S}])^{\mathcal{C}}_{n+1}$ from $(F, e_i[\mathcal{S}])^{\mathcal{C}}_{n+1}$ by*

- *If a variable* x *(resp. tag* $[u]$*) occurring in an element of* $(F, e_i[\mathcal{S}])^{\mathcal{C}}_{n+1}$ *has the corresponding variable* x *(resp. identifier* $(\_)_u$*) occurring in an element of* $(F, e_i[\mathcal{S}])^{\mathcal{C}}_l$ *for some* $l \leqslant n$ *that has been replaced with a fresh one* $x^\star$ *(resp.* $(\_)_{u^\star}$*) in the operation* $(F, e_i[\mathcal{S}])^{\mathcal{C}}_l \mapsto (F, e^\star_{i^\star}[\mathcal{S}])^{\mathcal{C}}_l$*, then replacing it with the one* $x^\star$ *(resp.* $[u^\star]$*);*
- *Replacing identifiers* $(\_)_v$ *occurring in elements of* $(F, e_i[\mathcal{S}])^{\mathcal{C}}_{n+1}$ *with fresh ones* $(\_)_{v^\star}$ *and then replacing the corresponding tags* $[v]$ *occurring in elements of* $(F, e_i[\mathcal{S}])^{\mathcal{C}}_l$ *for some* $l \leqslant n$*, if any, in the corresponding way* $[v] \mapsto [v^\star]$*;*

3. *Iterate the step 2 until* $(F, e_i[\mathcal{S}])^{\mathcal{C}}_{n+1} = \emptyset$*. The elements of* $(F, e_i[\mathcal{S}])^{\mathcal{C}} := \bigcup_{l=0}^{n} (F, e_i[\mathcal{S}])^{\mathcal{C}}_l$ *constitute a linear raw-term* $\delta : \Delta \dashv \Pi \vdash \phi : \Phi$ *that forms a substructure of* F*, and the elements of* $(F, e^\star_{i^\star}[\mathcal{S}])^{\mathcal{C}} := \bigcup_{i=0}^{n} (F, e^\star_{i^\star}[\mathcal{S}])^{\mathcal{C}}_l$ *constitute its copy* $\delta^\star : \Delta \dashv \Pi^\star \vdash \phi^\star : \Phi$ *modulo the choice of variables and numerals. Obtain a linear raw-term* $F\langle 2e_i[\mathcal{S}]/e_i[\mathcal{S}]\rangle$ *from* F *by replacing* δ*,* Π *and* ϕ *respectively with* $\delta \uplus \delta^\star$*,* $\Pi \uplus \Pi^\star$ *and* $\phi \uplus \phi^\star$ *in* F*.*

▶ Remark 14. The use of recursive deletion and copying is illustrated in Examples 17–19.

▶ **Notation 15.** *We use the letters* f*,* g*, etc. as meta-variables for the constructs* $e_i[\mathcal{S}]$*, and write* $\heartsuit\{\clubsuit/\spadesuit\}$ *for the* substitution *of* $\clubsuit$ *for* $\spadesuit$ *in* $\heartsuit$*. Given a (co)pattern* $\mathscr{M}$*, we write* $f(\mathscr{M})$*,* $g(\mathscr{M})$*, etc. for arbitrary constructs* $e_i[\mathcal{S}]$ *such that there is some* $e'_j[\mathcal{T}] \in \mathscr{M}$ *with* $j \in \mathcal{S}$*.*

▶ **Definition 16** (the $\ell\beta$-reduction)**.** *The* $\ell\beta$-reduction *is the binary relation* $\to_{\ell\beta} := \cup_{i=1}^{16} \to_{\ell\beta_i}$ *on linear raw-terms, where the ones* $\to_{\ell\beta_i}$ *are listed in Figure 4 in which we omit formulas.*

$$
\begin{array}{c}
\delta \dashv \Pi \uplus \mathrm{cut}(a_i[\mathcal{S}] \mid p_j[\mathcal{T}]) \vdash \phi \to_{\ell\beta_1} (\delta \dashv \Pi \uplus \mathrm{cut}(a_i \mid p_j[\mathcal{T}]) \vdash \phi)\{\mathcal{S} \cup j/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(a_i[\mathcal{S}] \mid p_j[\mathcal{T}]) \vdash \phi \to_{\ell\beta_2} (\delta \dashv \Pi \uplus \mathrm{cut}(a_i[\mathcal{S}] \mid p_j) \vdash \phi)\{\mathcal{T} \cup i/i\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \emptyset) \vdash \phi \to_{\ell\beta_3} (\delta \dashv \Pi \vdash \phi)\langle \emptyset/f(\mathscr{A})\rangle \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P} \uplus p[\mathcal{S}]) \vdash \phi \to_{\kappa\beta_4} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \mathrm{cut}(\mathscr{A} \mid p[\mathcal{S}]) \vdash \phi)\langle 2f(\mathscr{A})/f(\mathscr{A})\rangle \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \uplus a[\mathcal{S}] \mid \mathscr{P}) \vdash \phi \to_{\kappa\beta_5} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \mathrm{cut}(a[\mathcal{S}] \mid \mathscr{P}) \vdash \phi)\langle 2f(\mathscr{P})/f(\mathscr{P})\rangle \\
\delta \dashv \Pi \uplus \mathrm{cut}(a_i \mid \underline{x}_j) \vdash \phi \to_{\ell\beta_6} (\delta \dashv \Pi \vdash \phi)\{a/\overline{x}\}\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\overline{x}_i \mid p_j) \vdash \phi \to_{\ell\beta_7} (\delta \dashv \Pi \vdash \phi)\{p/\underline{x}\}\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\tau_i \mid \tau_j) \vdash \phi \to_{\ell\beta_8} (\delta \dashv \Pi \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\beta_i \mid \beta_j) \vdash \phi \to_{\ell\beta_9} (\delta \dashv \Pi \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \otimes_i \mathscr{B} \mid \mathscr{P} \otimes_j \mathscr{Q}) \vdash \phi \to_{\ell\beta_{10}} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \mathrm{cut}(\mathscr{B} \mid \mathscr{Q}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \parr_i \mathscr{B} \mid \mathscr{P} \parr_j \mathscr{Q}) \vdash \phi \to_{\ell\beta_{11}} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \mathrm{cut}(\mathscr{B} \mid \mathscr{Q}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \&_i \mid \pi^{\&}_{1,j}(\mathscr{P})) \vdash \phi \to_{\ell\beta_{12}} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\&_i \mathscr{B} \mid \pi^{\&}_{2,j}(\mathscr{Q})) \vdash \phi \to_{\ell\beta_{12}} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{B} \mid \mathscr{Q}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \&_i \mid \pi^{\&}_{2,j}(\mathscr{Q})) \vdash \phi \to_{\ell\beta_{12}} (\delta \dashv \Pi \vdash \phi)\langle \emptyset/f[+j]\rangle\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\&_i \mathscr{B} \mid \pi^{\&}_{1,j}(\mathscr{P})) \vdash \phi \to_{\ell\beta_{12}} (\delta \dashv \Pi \vdash \phi)\langle \emptyset/f[+j]\rangle\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\iota^{\oplus}_{1,i}(\mathscr{A}) \mid \mathscr{P} \oplus_j) \vdash \phi \to_{\ell\beta_{13}} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\iota^{\oplus}_{2,i}(\mathscr{B}), \oplus_j \mathscr{Q}) \vdash \phi \to_{\ell\beta_{13}} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{B} \mid \mathscr{Q}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\iota^{\oplus}_{2,i}(\mathscr{B}) \mid \mathscr{P} \oplus_j) \vdash \phi \to_{\ell\beta_{13}} (\delta \dashv \Pi \vdash \phi)\langle \emptyset/f[+j]\rangle\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\iota^{\oplus}_{1,i}(\mathscr{A}) \mid \oplus_j \mathscr{Q}) \vdash \phi \to_{\ell\beta_{13}} (\delta \dashv \Pi \vdash \phi)\langle \emptyset/f[+j]\rangle\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{P}^{\perp_i} \mid \mathscr{A}^{\perp_j}) \vdash \phi \to_{\ell\beta_{14}} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(!_i \mathscr{A} \mid !_j \mathscr{P}) \vdash \phi \to_{\ell\beta_{15}} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(?_i \mathscr{A} \mid ?_j \mathscr{P}) \vdash \phi \to_{\ell\beta_{16}} (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\}
\end{array}
$$

Figure 4 The $\ell\beta$-reduction

▶ **Example 17.** The linear σ-term

$$\dfrac{\dfrac{}{\underline{z}_7[8] : A \dashv \emptyset \vdash \overline{z}_8[7] : A}\,(\text{Ax}) \qquad \dfrac{\dfrac{\dfrac{}{\underline{x}_1[2] : A \dashv \emptyset \vdash \overline{x}_2[1] : A}\,(\text{Ax}) \qquad \dfrac{\dfrac{}{\dashv \emptyset \vdash \tau_{11} : \top}\,(\top\text{R})}{\emptyset : A \dashv \emptyset \vdash \tau_{11} : \top}\,(\text{WL})}{\underline{x}_1[2] : A \dashv \mathrm{cut}(\overline{x}_2[1] \mid \emptyset) \vdash \tau_{11} : \top}\,(\textsc{Cut}) \qquad \dfrac{}{\underline{y}_3[4] : A \dashv \emptyset \vdash \overline{y}_4[3] : A}\,(\text{Ax})}{\underline{x}_1[2,5], \underline{y}_3[4,6] : A \dashv \mathrm{cut}(\overline{x}_2[1] \mid \emptyset)[5] \vdash \tau_{11}\&_5, \&_6(\overline{y}_4[3]) : \top \,\&\, A}\,(\&\text{R})}{\underline{z}_7[8] : A \dashv \mathrm{cut}(\overline{z}_8[7] \mid \underline{x}_1[2,5], \underline{y}_3[4,6]), \mathrm{cut}(\overline{x}_2[1] \mid \emptyset)[5] \vdash \tau_{11}\&_5, \&_6(\overline{y}_4[3]) : \top \,\&\, A}\,(\textsc{Cut})$$

has the chain of the $\ell\beta$-reduction

$$\begin{aligned}
&\underline{z}_7[8] : A \dashv \mathrm{cut}(\overline{z}_8[7] \mid \underline{x}_1[2,5], \underline{y}_3[4,6]), \mathrm{cut}(\overline{x}_2[1] \mid \emptyset)[5] \vdash \tau_{11}\&_5, \&_6(\overline{y}_4[3]) : \top \,\&\, A\\
\to_{\ell\beta}\ &\underline{z}_7[8] : A \dashv \mathrm{cut}(\overline{z}_8[7] \mid \underline{y}_3[4,6]) \vdash \tau_{11}\&_5, \&_6(\overline{y}_4[3]) : \top \,\&\, A\\
\to_{\ell\beta}\ &\underline{z}_7[8] : A \dashv \mathrm{cut}(\overline{z}_8 \mid \underline{y}_3[4,6]) \vdash \tau_{11}\&_5, \&_6(\overline{y}_4[3,7]) : \top \,\&\, A\\
\to_{\ell\beta}\ &\underline{z}_7[4,6,8] : A \dashv \mathrm{cut}(\overline{z}_8 \mid \underline{y}_3) \vdash \tau_{11}\&_5, \&_6(\overline{y}_4[3,7]) : \top \,\&\, A\\
\to_{\ell\beta}\ &\underline{y}_7[4,6] : A \dashv \emptyset \vdash \tau_{11}\&_5, \&_6(\overline{y}_4[7]) : \top \,\&\, A.
\end{aligned}$$

▶ **Example 18.** The linear σ-term

$$\dfrac{\dfrac{}{\underline{z}_5[6] : !A \dashv \emptyset \vdash \overline{z}_6[5] : !A}\,(\text{Ax}) \qquad \dfrac{\dfrac{\dfrac{}{\underline{x}_1[2] : !A \dashv \emptyset \vdash \overline{x}_2[1] : !A}\,(\text{Ax}) \qquad \dfrac{}{\underline{y}_3[4] : !A \dashv \emptyset \vdash \overline{y}_4[3] : !A}\,(\text{Ax})}{\underline{x}_1[2,5] : !A, \underline{y}_3[4,5] : !A \dashv \emptyset \vdash \overline{x}_2[1] \otimes_5 \overline{y}_4[3] : !A \otimes !A}\,(\otimes\text{R})}{\underline{x}_1[2,7], \underline{y}_3[4,7] : !A \dashv \emptyset \vdash \overline{x}_2[1] \otimes_7 \overline{y}_4[3] : !A \otimes !A}\,(!\text{C})}{\underline{z}_5[6] : !A \dashv \mathrm{cut}(\overline{z}_6[5] \mid \underline{x}_1[2,7], \underline{y}_3[4,7]) \vdash \overline{x}_2[1] \otimes_7 \overline{y}_4[3] : !A \otimes !A}\,(\textsc{Cut})$$

has the chain of the $\ell\beta$-reduction

$$\begin{aligned}
&\underline{z}_5[6] : !A \dashv \mathrm{cut}(\overline{z}_6[5] \mid \underline{x}_1[2,7], \underline{y}_3[4,7]) \vdash \overline{x}_2[1] \otimes_7 \overline{y}_4[3] : !A \otimes !A\\
\to_{\ell\beta}\ &\underline{z}_5[6], \underline{z'}_8[9] : !A \dashv \mathrm{cut}(\overline{z}_6[5] \mid \underline{x}_1[2,7]), \mathrm{cut}(\overline{z'}_9[8] \mid \underline{y}_3[4,7]) \vdash \overline{x}_2[1] \otimes_7 \overline{y}_4[3] : !A \otimes !A\\
\to^*_{\ell\beta}\ &\underline{z}_5[6,2,7], \underline{z'}_8[9,4,7] : !A \dashv \mathrm{cut}(\overline{z}_6 \mid \underline{x}_1), \mathrm{cut}(\overline{z'}_9 \mid \underline{y}_3) \vdash \overline{x}_2[1,5] \otimes_7 \overline{y}_4[3,8] : !A \otimes !A\\
\to^*_{\ell\beta}\ &\underline{z}_5[2,7], \underline{y}_8[4,7] : !A \dashv \emptyset \vdash \overline{z}_2[5] \otimes_7 \overline{y}_4[8] : !A \otimes !A.
\end{aligned}$$

▶ **Example 19.** We have the following $\ell\beta$-reduction on the linear σ-term derived in Figure 3:

$$\begin{aligned}
&\pi_{1,13}(\underline{z}_{11}[12])[17], \pi_{1,14}(\underline{w}_{15}[16])[18] \dashv \mathrm{cut}(\overline{z}_{12}[11,13]\&_{17}, \&_{18}(\overline{w}_{16}[15,14]) \mid \pi^{\&}_{1,5}(\underline{x}_1[2])[7,9], \pi^{\&}_{2,6}(\underline{y}_3[4])[8,9])\\
&\quad \vdash \iota^{\oplus}_{1,9}(\overline{x}_2[1,5]\&_7, \&_8\overline{y}_4[3,6])\\
\to^*_{\ell\beta}\ &\pi_{1,13}(\underline{z}_{11}[12])[17], \pi_{1,14}(\underline{w}_{15}[16])[18] \dashv \mathrm{cut}(\overline{z}_{12}[11,13]\&_{17} \mid \pi^{\&}_{1,5}(\underline{x}_1[2])[7,9], \pi^{\&}_{2,6}(\underline{y}_3[4])[8,9]),\\
&\quad \mathrm{cut}(\&_{18}(\overline{w}_{16}[15,14]) \mid \pi^{\&}_{1,23}(\underline{x}_{19}[20])[7,9], \pi^{\&}_{2,24}(\underline{y}_{21}[22])[8,9]) \vdash \iota^{\oplus}_{1,9}((\overline{x}_2[1,5], \overline{x}_{20}[19,23])\&_7, \&_8(\overline{y}_4[3,6], \overline{y}_{22}[21,24]))\\
\to^*_{\ell\beta}\ &\pi_{1,13}(\underline{z}_{11}[12])[17], \pi_{1,31}(\underline{z}_{29}[30])[35], \pi_{1,14}(\underline{w}_{15}[16])[18], \pi_{1,32}(\underline{w}_{33}[34])[36] \dashv \mathrm{cut}(\overline{z}_{12}[11,13]\&_{17} \mid \pi^{\&}_{1,5}(\underline{x}_1[2])[7,9]),\\
&\quad \mathrm{cut}(\overline{z}_{30}[29,31]\&_{35} \mid \pi^{\&}_{2,6}(\underline{y}_3[4])[8,9]), \mathrm{cut}(\&_{18}(\overline{w}_{16}[15,14]) \mid \pi^{\&}_{1,23}(\underline{x}_{19}[20])[7,9]), \mathrm{cut}(\&_{36}(\overline{w}_{34}[33,32]) \mid \pi^{\&}_{2,24}(\underline{y}_{21}[22])[8,9])\\
&\quad \vdash \iota^{\oplus}_{1,9}((\overline{x}_2[1,5], \overline{x}_{20}[19,23])\&_7, \&_8(\overline{y}_4[3,6], \overline{y}_{22}[21,24]))\\
\to^*_{\ell\beta}\ &\pi_{1,13}(\underline{z}_{11}[12])[17], \pi_{1,32}(\underline{w}_{33}[34])[36] \dashv \mathrm{cut}(\overline{z}_{12}[11,13]\&_{17} \mid \pi^{\&}_{1,5}(\underline{x}_1[2])[7,9]), \mathrm{cut}(\&_{36}(\overline{w}_{34}[33,32]) \mid \pi^{\&}_{2,24}(\underline{y}_{21}[22])[8,9])\\
&\quad \vdash \iota^{\oplus}_{1,9}(\overline{x}_2[1,5]\&_7, \&_8(\overline{y}_{22}[21,24]))\\
\to^*_{\ell\beta}\ &\pi_{1,13}(\underline{z}_{11}[12])[17,7,9], \pi_{1,32}(\underline{w}_{33}[34])[36,8,9] \dashv \mathrm{cut}(\overline{z}_{12}[11,13]\&_{17} \mid \pi^{\&}_{1,5}(\underline{x}_1[2])), \mathrm{cut}(\&_{36}(\overline{w}_{34}[33,32]) \mid \pi^{\&}_{2,24}(\underline{y}_{21}[22]))\\
&\quad \vdash \iota^{\oplus}_{1,9}(\overline{x}_2[1,5]\&_7, \&_8(\overline{y}_{22}[21,24]))\\
\to^*_{\ell\beta}\ &\pi_{1,13}(\underline{z}_{11}[12])[7,9], \pi_{1,32}(\underline{w}_{33}[34])[8,9] \dashv \mathrm{cut}(\overline{z}_{12}[11,13] \mid \underline{x}_1[2]), \mathrm{cut}(\overline{w}_{34}[33,32] \mid \underline{y}_{21}[22]) \vdash \iota^{\oplus}_{1,9}(\overline{x}_2[1]\&_7, \&_8(\overline{y}_{22}[21]))\\
\to^*_{\ell\beta}\ &\pi_{1,13}(\underline{z}_{11}[12,2])[7,9], \pi_{1,32}(\underline{w}_{33}[34,22])[8,9] \dashv \mathrm{cut}(\overline{z}_{12} \mid \underline{x}_1), \mathrm{cut}(\overline{w}_{34} \mid \underline{y}_{21}) \vdash \iota^{\oplus}_{1,9}(\overline{x}_2[1,11,13]\&_7, \&_8(\overline{y}_{22}[21,33,32]))\\
\to^*_{\ell\beta}\ &\pi_{1,13}(\underline{x}_{11}[2])[7,9], \pi_{1,32}(\underline{y}_{33}[22])[8,9] \dashv \emptyset \vdash \iota^{\oplus}_{1,9}(\overline{x}_2[11,13]\&_7, \&_8(\overline{y}_{22}[33,32]))
\end{aligned}$$

where we omit types for lack of space.

We now turn to proving basic properties of the $\ell\beta$-reduction:

▶ **Theorem 20** (linear subject reduction). *If F is a linear σ-term from Δ to Φ that has the $\ell\beta$-reduction $F \to_{\ell\beta} F'$, then F' is also a linear σ-term from Δ to Φ.*

Proof. Based on the cut-elimination on the sequent calculus for linear logic [18, §21]. ◀

▶ **Lemma 21** (linear diamond property)**.** *If F is a linear σ-program with $F \to_{\ell\beta} F'$, $F \to_{\ell\beta} F''$ and $F' \neq F''$, then there is another linear σ-term F''' such that $F' \to_{\ell\beta} F'''$ and $F'' \to_{\ell\beta} F'''$.*

Proof. By a straightforward yet lengthy case analysis on $F \to_{\ell\beta} F'$ and $F \to_{\ell\beta} F''$. ◀

▶ **Theorem 22** (linear confluence)**.** *If F is a linear σ-term such that $F \to^*_{\ell\beta} F'$ and $F \to^*_{\ell\beta} F''$, then there is another linear σ-term F''' such that $F' \to^*_{\ell\beta} F'''$ and $F'' \to^*_{\ell\beta} f'''$.*

Proof. Immediate from Lemma 21. ◀

▶ **Definition 23** (sizes of states)**.** *The size $|\mathscr{M}|$ of a (co)pattern $\mathscr{M}$ is defined inductively by*

$$|\emptyset| := 0 \qquad |\mathscr{M} \uplus e_i[\mathcal{S}]| := |\mathscr{M}| + |e_i[\mathcal{S}]| + 1 \qquad |e_i[\mathcal{S}]| := |e_i| + |\mathcal{S}| + 1$$

$$|\underline{x}_i| := |\overline{x}_i| := |\beta_i| := |\tau_i| := 1 \qquad |\mathscr{M} \otimes_i \mathscr{N}| := |\mathscr{M} ⅋_i \mathscr{N}| := |\mathscr{M}| + |\mathscr{N}| + 1$$

$$|\pi^{\&}_{j,i}(\mathscr{M})| := |\mathscr{M}\&_i| := |\&_i\mathscr{M}| := |\iota^{\oplus}_{j,i}(\mathscr{M})| := |\mathscr{M} \oplus_i| := |\oplus_i \mathscr{M}| := |\mathscr{M}^{\perp_i}| := |!_i\mathscr{M}| := |?_i\mathscr{M}| := |\mathscr{M}| + 1$$

where $|\mathcal{S}|$ is the number of elements of the synchronisation $\mathcal{S}$. The size $|\mathrm{cut}(\mathscr{A}, \mathscr{P})|$ of a cut $\mathrm{cut}(\mathscr{A}, \mathscr{P})$ and the size $|\Pi|$ of a state Π are then defined by

$$|\mathrm{cut}(\mathscr{A}, \mathscr{P})| := (|\mathscr{A}| \times |\mathscr{P}|) + 1 \qquad |\Pi| := \sum_{\sigma \in \Pi} |\sigma|.$$

▶ **Theorem 24** (linear strong normalisation)**.** *Every sequence of the $\ell\beta$-reductions on linear σ-terms is finite.*

Proof. Assume that there is an infinite sequence $F_0 \to_{\ell\beta} F_1 \to_{\ell\beta} F_2 \to_{\ell\beta} \dots$ of linear σ-terms F_i with states Π_i ($i \in \mathbb{N}$). Because each $\ell\beta$-reduction decreases the sizes of states of linear σ-terms, we have $|\Pi_0| > |\Pi_1| > |\Pi_2| > \dots$, which contradicts the finiteness of $|\Pi_0|$. ◀

The proofs of Theorems 22 and 24 are quite straightforward thanks to the simplicity of the $\ell\beta$-reduction, in contrast with the (linear) λ-calculus, let alone the (linear) $\lambda\mu$-calculus.

3 Term Calculi for Classical and Intuitionistic Logics

Now, it is just a routine to apply the format of the linear σ-calculus (§2) to classical logic:

▶ **Definition 25** (the classical σ-calculus)**.** *The* classical σ-calculus *consists of the following:*

- (Types) Types A, B, *etc. are formulas of classical logic [17], i.e., the expressions defined by the grammar*

  $$A, B := X \mid \mathrm{tt} \mid \mathrm{ff} \mid A \wedge B \mid A \vee B \mid A \Rightarrow B$$

  where X ranges over $\mathcal{P}$, we define $\neg A := A \Rightarrow \mathrm{ff}$, and we call tt truth, ff falsity, $\wedge$ conjunction, $\vee$ disjunction, $\Rightarrow$ implication, *and* $\neg$ negation.
- (Patterns and copatterns) Patterns $\mathscr{P}$, $\mathscr{Q}$, *etc. and* copatterns $\mathscr{A}$, $\mathscr{B}$, *etc. are the finite multisets defined by the grammar*

  $$p_i, q_i := \underline{x}_i \mid \vartheta_i \mid \varrho_i \mid \pi^{\wedge}_{j,i}(\mathscr{P}) \mid \mathscr{P}\vee_i \mid \vee_i\mathscr{Q} \mid \lambda_i(\mathscr{A}, \mathscr{Q})$$
  $$a_i, b_i := \overline{x}_i \mid \vartheta_i \mid \varrho_i \mid \mathscr{A}\wedge_i \mid \wedge_i\mathscr{B} \mid \iota^{\vee}_{j,i}(\mathscr{A}) \mid \lambda_i(\mathscr{P}, \mathscr{B})$$
  $$\mathscr{P}, \mathscr{Q} := \emptyset \mid \mathscr{P} \uplus \mathscr{Q} \mid \{p_i[\emptyset]\} \mid \mathscr{P}[+i]$$
  $$\mathscr{A}, \mathscr{B} := \emptyset \mid \mathscr{A} \uplus \mathscr{B} \mid \{a_i[\emptyset]\} \mid \mathscr{A}[+i]$$

  where x ranges over $\mathcal{T}$, i over $\mathbb{N}$, and j over $\{1, 2\}$. (Co)contexts, states *and* (classical) raw-terms *are as in Definition 5 except that* superpositions *are with respect to $\wedge$ or $\vee$.*

- (Terms) Classical σ-terms *from* Δ *to* Φ *are classical raw-terms of the form* $\delta : \Delta \dashv \Pi \vdash \phi : \Phi$ *derivable by the typing rules in Figure 5.*

$$(\mathrm{XL})\ \frac{\delta : \Delta, \mathscr{P} : A, \mathscr{Q} : B, \gamma : \Gamma \dashv \Pi \vdash \phi : \Phi}{\delta : \Delta, \mathscr{Q} : B, \mathscr{P} : A, \gamma : \Gamma \dashv \Pi \vdash \phi : \Phi} \qquad (\mathrm{XR})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi, \mathscr{A} : A, \mathscr{B} : B, \psi : \Psi}{\delta : \Delta \dashv \Pi \vdash \phi : \Phi, \mathscr{B} : B, \mathscr{A} : A, \psi : \Psi}$$

$$(\mathrm{WL})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}{\delta : \Delta, \emptyset : A \dashv \Pi \vdash \phi : \Phi} \qquad (\mathrm{WR})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}{\delta : \Delta \dashv \Pi \vdash \emptyset : A, \phi : \Phi}$$

$$(\mathrm{CL})\ \frac{\delta : \Delta, \mathscr{P} : A, \mathscr{P}' : A \dashv \Pi \vdash \phi : \Phi}{\delta : \Delta, \mathscr{P} \uplus \mathscr{P}' : A \dashv \Pi \vdash \phi : \Phi} \qquad (\mathrm{CR})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \mathscr{A}' : A, \phi : \Phi}{\delta : \Delta \dashv \Pi \vdash \mathscr{A} \uplus \mathscr{A}' : A, \phi : \Phi}$$

$$(\textsc{Cut})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi \qquad \gamma : \Gamma, \mathscr{P} : A \dashv \Sigma \vdash \psi : \Psi}{\delta : \Delta, \gamma : \Gamma \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \Sigma \vdash \phi : \Phi, \psi : \Psi}\ (\sharp\mathscr{A} = 1)$$

$$(\mathrm{Ax})\ \frac{}{\underline{x}_i[j] : A \dashv \emptyset \vdash \overline{x}_j[i] : A} \qquad (\mathrm{ttL})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}{\delta[+i] : \Delta, \vartheta_i : \mathrm{tt} \dashv \Pi[+i] \vdash \phi[+i] : \Phi} \qquad (\mathrm{ttR})\ \frac{}{\dashv \emptyset \vdash \vartheta_i : \mathrm{tt}}$$

$$(\mathrm{ffL})\ \frac{}{\varrho_i : \mathrm{ff} \dashv \emptyset \vdash} \qquad (\mathrm{ffR})\ \frac{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash \varrho_i : \mathrm{ff}, \phi[+i] : \Phi}$$

$$(\wedge\mathrm{L})\ \frac{\delta : \Delta, \mathscr{P} : A_j \dashv \Pi \vdash \phi : \Phi}{\delta[+i] : \Delta, \pi^{\wedge}_{j,i}(\mathscr{P}) : A_1 \wedge A_2 \dashv \Pi[+i] \vdash \phi[+i] : \Phi}\ (j \in \{1, 2\})$$

$$(\wedge\mathrm{R})\ \frac{\delta_1 : \Delta \dashv \Pi_1 \vdash \mathscr{A}_1 : A_1, \phi_1 : \Phi \qquad \delta_2 : \Delta \dashv \Pi_2 \vdash \mathscr{A}_2 : A_2, \phi_2 : \Phi}{\delta_1[+i] \uplus \delta_2[+j] : \Delta \dashv \Pi_1[+i] \uplus \Pi_2[+j] \vdash \mathscr{A}_1 \wedge_i, \wedge_j \mathscr{A}_2 : A_1 \wedge A_2, \phi_1[+i] \uplus \phi_2[+j] : \Phi}$$

$$(\vee\mathrm{L})\ \frac{\delta_1 : \Delta, \mathscr{P}_1 : A_1 \dashv \Pi_1 \vdash \phi_1 : \Phi \qquad \delta_2 : \Delta, \mathscr{P}_2 : A_2 \dashv \Pi_2 \vdash \phi_2 : \Phi}{\delta_1[+i] \uplus \delta_2[+j] : \Delta, \mathscr{P}_1 \vee_i, \vee_j \mathscr{P}_2 : A_1 \vee A_2 \dashv \Pi_1[+i] \uplus \Pi_2[+j] \vdash \phi_1[+i] \uplus \phi_2[+j] : \Phi}$$

$$(\vee\mathrm{R})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A_j, \phi : \Phi}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash \iota^{\vee}_{j,i}(\mathscr{A}) : A_1 \vee A_2, \phi[+i] : \Phi}\ (j \in \{1, 2\})$$

$$(\Rightarrow\mathrm{L})\ \frac{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi \qquad \gamma : \Gamma, \mathscr{Q} : B \dashv \Sigma \vdash \psi : \Psi}{\delta[+i] : \Delta, \gamma[+i] : \Gamma, \lambda_i(\mathscr{A}, \mathscr{Q}) : A \Rightarrow B \dashv \Pi[+i] \uplus \Sigma[+i] \vdash \phi[+i] : \Phi, \psi[+i] : \Psi}$$

$$(\Rightarrow\mathrm{R})\ \frac{\delta : \Delta, \mathscr{P} : A \dashv \Pi \vdash \mathscr{B} : B, \phi : \Phi}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash \lambda_i(\mathscr{P}, \mathscr{B}) : A \Rightarrow B, \phi[+i] : \Phi}$$

Figure 5 The typing rules of the classical σ-calculus

Because the typing rules of the classical σ-calculus correspond precisely to the derivation rules of the sequent calculus for classical logic [8, 17], we have:

▶ **Proposition 26** (a CHI for classical logic)**.** *A sequent* $\Delta \vdash \Phi$ *is derivable in the sequent calculus for classical logic if and only if there is a classical* σ*-term* $\delta : \Delta \dashv \Pi \vdash \phi : \Phi$ *for some* δ*,* Π *and* ϕ*, where the derivation has no cut if and only if* $\Pi = \emptyset$*.*

The classical σ-calculus is closed under substitution just like the case of the linear σ-calculus (Theorem 11). We leave the details to the reader.

▶ **Definition 27** (the $\kappa\beta$-reduction)**.** *The* $\kappa\beta$-reduction *is the binary relation* $\rightarrow_{\kappa\beta} := \cup_{i=1}^{12} \rightarrow_{\kappa\beta_i}$ *on classical raw-terms, where the binary relations* $\rightarrow_{\kappa\beta_i}$ *are listed in Figure 6.*

As in the case of the $\ell\beta$-reduction, we can show basic properties of the $\kappa\beta$-reduction:

$$
\begin{array}{rl}
\delta \dashv \Pi \uplus \mathrm{cut}(a_i[\mathcal{S}] \mid p_j[\mathcal{T}]) \vdash \phi \rightarrow_{\kappa\beta_1} & (\delta \dashv \Pi \uplus \mathrm{cut}(a_i \mid p_j[\mathcal{T}]) \vdash \phi)\{\mathcal{S} \cup j/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(a_i[\mathcal{S}] \mid p_j[\mathcal{T}]) \vdash \phi \rightarrow_{\kappa\beta_2} & (\delta \dashv \Pi \uplus \mathrm{cut}(a_i[\mathcal{S}] \mid p_j) \vdash \phi)\{\mathcal{T} \cup i/i\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \emptyset) \vdash \phi \rightarrow_{\kappa\beta_3} & (\delta \dashv \Pi \vdash \phi)\langle \emptyset / f(\mathscr{A})\rangle \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P} \uplus p[\mathcal{S}]) \vdash \phi \rightarrow_{\kappa\beta_4} & (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \mathrm{cut}(\mathscr{A} \mid p[\mathcal{S}]) \vdash \phi)\langle 2f(\mathscr{A})/f(\mathscr{A})\rangle \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \uplus a[\mathcal{S}] \mid \mathscr{P}) \vdash \phi \rightarrow_{\kappa\beta_5} & (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \mathrm{cut}(a[\mathcal{S}] \mid \mathscr{P}) \vdash \phi)\langle 2f(\mathscr{P})/f(\mathscr{P})\rangle \\
\delta \dashv \Pi \uplus \mathrm{cut}(a_i \mid \underline{x}_j) \vdash \phi \rightarrow_{\kappa\beta_6} & (\delta \dashv \Pi \vdash \phi)\{a/\overline{x}\}\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\overline{x}_i \mid p_j) \vdash \phi \rightarrow_{\kappa\beta_7} & (\delta \dashv \Pi \vdash \phi)\{p/\underline{x}\}\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\vartheta_i \mid \vartheta_j) \vdash \phi \rightarrow_{\kappa\beta_8} & (\delta \dashv \Pi \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\varrho_i \mid \varrho_j) \vdash \phi \rightarrow_{\kappa\beta_9} & (\delta \dashv \Pi \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \wedge_i \mid \pi^{\wedge}_{1,j}(\mathscr{P})) \vdash \phi \rightarrow_{\kappa\beta_{10}} & (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\wedge_i \mathscr{B} \mid \pi^{\wedge}_{2,j}(\mathscr{Q})) \vdash \phi \rightarrow_{\kappa\beta_{10}} & (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{B} \mid \mathscr{Q}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \wedge_i \mid \pi^{\wedge}_{2,j}(\mathscr{Q})) \vdash \phi \rightarrow_{\kappa\beta_{10}} & (\delta \dashv \Pi \vdash \phi)\langle \emptyset / f[+j]\rangle\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\wedge_i \mathscr{B} \mid \pi^{\wedge}_{1,j}(\mathscr{P})) \vdash \phi \rightarrow_{\kappa\beta_{10}} & (\delta \dashv \Pi \vdash \phi)\langle \emptyset / f[+j]\rangle\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\iota^{\vee}_{1,i}(\mathscr{A}) \mid \mathscr{P} \vee_j) \vdash \phi \rightarrow_{\kappa\beta_{11}} & (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\iota^{\vee}_{2,i}(\mathscr{B}), \vee_j \mathscr{Q}) \vdash \phi \rightarrow_{\kappa\beta_{11}} & (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{B} \mid \mathscr{Q}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\iota^{\vee}_{2,i}(\mathscr{B}) \mid \mathscr{P} \vee_j) \vdash \phi \rightarrow_{\kappa\beta_{11}} & (\delta \dashv \Pi \vdash \phi)\langle \emptyset / f[+j]\rangle\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\iota^{\vee}_{1,i}(\mathscr{A}) \mid \vee_j \mathscr{Q}) \vdash \phi \rightarrow_{\kappa\beta_{11}} & (\delta \dashv \Pi \vdash \phi)\langle \emptyset / f[+j]\rangle\{\emptyset/i\}\{\emptyset/j\} \\
\delta \dashv \Pi \uplus \mathrm{cut}(\lambda_i(\mathscr{P}, \mathscr{B}) \mid \lambda_j(\mathscr{A}, \mathscr{Q})) \vdash \phi \rightarrow_{\kappa\beta_{12}} & (\delta \dashv \Pi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \mathrm{cut}(\mathscr{B} \mid \mathscr{Q}) \vdash \phi)\{\emptyset/i\}\{\emptyset/j\}
\end{array}
$$

Figure 6 The $\kappa\beta$-reduction

▶ **Theorem 28** (basic properties of the $\kappa\beta$-reduction)**.** *The $\kappa\beta$-reduction enjoys subject reduction, diamond property, confluence and strong normalisation.*

Finally, we carve out our term calculus for intuitionistic logic from the classical σ-calculus by the standard method:

▶ **Definition 29** (the intuitionistic σ-calculus)**.** *The* intuitionistic σ-calculus *is obtained from the classical σ-calculus by restricting raw-terms to those $\delta : \Delta \dashv \Pi \vdash \phi : \Phi$ with the length of Φ at most one. We call these raw-terms derivable by the typing rules* intuitionistic σ-terms*.*

▶ **Proposition 30** (a CHI for intuitionistic logic)**.** *A sequent $\Delta \vdash B$ is derivable in the sequent calculus for intuitionistic logic if and only if there is an intuitionistic σ-term $\delta : \Delta \dashv \Pi \vdash \mathscr{B} : B$ for some δ, Π and $\mathscr{B}$, where the derivation has no cut if and only if $\Pi = \emptyset$.*

▶ **Definition 31** (the $\jmath\beta$-reduction)**.** *The* $\jmath\beta$-reduction *is the binary relation $\rightarrow_{\jmath\beta}$ on intuitionistic raw-terms obtained from the $\kappa\beta$-reduction by restricting it to intuitionistic raw-terms.*

▶ **Theorem 32** (basic properties of the $\jmath\beta$-reduction)**.** *The $\jmath\beta$-reduction $\rightarrow_{\jmath\beta}$ enjoys subject reduction, diamond property, confluence and strong normalisation.*

4 Conclusion and Future Work

We have introduced two-sided term calculi equipped with reductions for classical, intuitionistic and linear logics, respectively. We have also shown that these term calculi and reductions enjoy basic computational properties. Notably, our term calculi dispense with conversions, i.e., overcome one of the fundamental problems in proof theory.

For future work, we shall extend the term calculi by programming constructs such as booleans, natural numbers and recursion. Moreover, we plan to lift the two-sided format to dependent type theories for combining dependent types and classical/linear reasoning.

References

1. Samson Abramsky. Computational interpretations of linear logic. *Theoretical computer science*, 111(1-2):3–57, 1993.
2. Andrew Barber and Gordon Plotkin. *Dual intuitionistic linear logic*. University of Edinburgh, Department of Computer Science, Laboratory for . . . , 1996.
3. Nick Benton, Gavin Bierman, Valeria De Paiva, and Martin Hyland. A term calculus for intuitionistic linear logic. In *International Conference on Typed Lambda Calculi and Applications*, pages 75–90. Springer, 1993.
4. Gavin M. Bierman. A note on full intuitionistic linear logic. *Annals of Pure and Applied Logic*, 79(3):281–287, 1996.
5. Gavin M Bierman. A classical linear λ-calculus. *Theoretical Computer Science*, 227(1-2):43–78, 1999.
6. Alonzo Church. A formulation of the simple theory of types. *The journal of symbolic logic*, 5(2):56–68, 1940.
7. Pierre-Louis Curien and Hugo Herbelin. The duality of computation. *ACM sigplan notices*, 35(9):233–243, 2000.
8. Gerhard Gentzen. Untersuchungen über das logische schließen. i. *Mathematische zeitschrift*, 39(1):176–210, 1935.
9. Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–101, 1987.
10. Jean-Yves Girard. Towards a geometry of interaction. *Contemporary Mathematics*, 92(69-108):6, 1989.
11. Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and types*, volume 7. Cambridge university press Cambridge, 1989.
12. Dominic JD Hughes and Rob J Van Glabbeek. Proof nets for unit-free multiplicative-additive linear logic. In *LICS*, pages 1–10, 2003.
13. Michel Parigot. Free deduction: An analysis of "computations" in classical logic. In *Logic Programming*, pages 361–380. Springer, 1992.
14. Michel Parigot. $\lambda\mu$-calculus: an algorithmic interpretation of classical natural deduction. In *International Conference on Logic for Programming Artificial Intelligence and Reasoning*, pages 190–201. Springer, 1992.
15. Peter Selinger. Control categories and duality: on the categorical semantics of the lambda-mu calculus. *Mathematical Structures in Computer Science*, 11(2):207–260, 2001.
16. Morten Heine Sørensen and Pawel Urzyczyn. *Lectures on the Curry-Howard isomorphism*. Elsevier, 2006.
17. Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic proof theory*. Number 43. Cambridge University Press, 2000.
18. AS TROELSTRA. Lectures on linear logic. *CSLI Lecture Notes 29*, 3, 1992.
19. Philip Wadler. Call-by-name is dual to call-by-value. In *International Conference on Functional Programming*, 2003.
20. Philip Wadler. Down with the bureaucracy of syntax! pattern matching for classical linear logic. *Unpublished draft. Google: wadler+ dual*, 2004.
21. Norihiro Yamada. Sequent calculi for a unity of logic. *arXiv preprint arXiv:2001.06138*, 2020.

A Proof of Theorem 20

We provide a proof of Theorem 20 based on a cut-elimination algorithm for linear logic. For brevity, we omit the exchange rules XR and XL (Figure 1) from derivation trees.

The proof idea is to show that for each $\ell\beta$-reduction there is a corresponding cut-elimination on derivation trees. In the following, we list all the possible cases.

A.1 Principal Cuts

The first pattern is an application of the rule Cut (Figure 1) whose cut formulas are both the principal formulas of the rules at the end of the two hypotheses.

- ($\top$R, $\top$L)-Cut. A derivation tree of the form

$$\text{(Cut)}\ \dfrac{\text{($\top$R)}\ \dfrac{}{\dashv\vdash \tau_i : \top} \qquad \text{($\top$L)}\ \dfrac{\overset{u}{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}}{\delta[+j] : \Delta, \tau_j : \top \dashv \Pi[+j] \vdash \phi[+j] : \Phi}}{\delta[+j] : \Delta \dashv \Pi[+j] \uplus \mathrm{cut}(\tau_i \mid \tau_j) \vdash \phi[+j] : \Phi}$$

  is transformed into the derivation tree

$$\overset{u}{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}$$

- ($\bot$R, $\bot$L)-Cut. A derivation tree of the form

$$\text{(Cut)}\ \dfrac{\text{($\bot$R)}\ \dfrac{\overset{u}{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash \beta_i : \bot, \phi[+i] : \Phi} \qquad \text{($\bot$L)}\ \dfrac{}{\beta_j : \bot \dashv\vdash}}{\delta[+i] : \Delta \dashv \Pi[+i] \uplus \mathrm{cut}(\beta_i \mid \beta_j) \vdash \phi[+i] : \Phi}$$

  is transformed into the derivation tree

$$\overset{u}{\delta : \Delta \dashv \Pi \vdash \phi : \Phi}$$

- $((_)^{\bot}$R, $(_)^{\bot}$L)-Cut. A derivation tree of the form

$$\text{(Cut)}\ \dfrac{\text{($(\_)^{\bot}$R)}\ \dfrac{\overset{u}{\delta : \Delta, \mathscr{P} : A \dashv \Pi \vdash \phi : \Phi}}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash \mathscr{P}^{\bot i} : A^{\bot}, \phi[+i] : \Phi} \qquad \text{($(\_)^{\bot}$L)}\ \dfrac{\overset{v}{\gamma : \Gamma \dashv \Sigma \vdash \mathscr{A} : A, \psi : \Psi}}{\gamma[+j] : \Gamma, \mathscr{A}^{\bot j} : A^{\bot} \dashv \Sigma[+j] \vdash \psi[+j] : \Psi}}{\delta[+i] : \Delta, \gamma[+j] : \Gamma \dashv \Pi[+i] \uplus \Sigma[+j] \uplus \mathrm{cut}(\mathscr{P}^{\bot i} \mid \mathscr{A}^{\bot j}) \vdash \phi[+i] : \Phi, \psi[+j] : \Psi}$$

  is transformed into the derivation tree

$$\text{(Cut)}\ \dfrac{\overset{v}{\gamma : \Gamma \dashv \Sigma \vdash \mathscr{A} : A, \psi : \Psi} \qquad \overset{u}{\delta : \Delta, \mathscr{P} : A \dashv \Pi \vdash \phi : \Phi}}{\delta : \Delta, \gamma : \Gamma \dashv \Pi \uplus \Sigma \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \vdash \phi : \Phi, \psi : \Psi}$$

- ($\otimes$R, $\otimes$L)-Cut. A derivation tree of the form

$$\text{(Cut)}\ \dfrac{\text{($\otimes$R)}\ \dfrac{\overset{u}{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi} \qquad \overset{v}{\gamma : \Gamma \dashv \Sigma \vdash \mathscr{B} : B, \psi : \Psi}}{\delta[+i] : \Delta, \gamma[+i] : \Gamma \dashv \Pi[+i] \uplus \Sigma[+i] \vdash \mathscr{A} \otimes_i \mathscr{B} : A \otimes B, \phi[+i] : \Phi, \psi[+i] : \Psi} \qquad \text{($\otimes$L)}\ \dfrac{\overset{w}{\theta : \Theta, \mathscr{P} : A, \mathscr{Q} : B \dashv \Xi \vdash \varphi : \Upsilon}}{\theta[+j] : \Theta, \mathscr{P} \otimes_j \mathscr{Q} : A \otimes B \dashv \Xi[+j] \vdash \varphi[+j] : \Upsilon}}{\delta[+i] : \Delta, \gamma[+i] : \Gamma, \theta[+j] : \Theta \dashv \Pi[+i] \uplus \Sigma[+i] \uplus \Xi[+j] \uplus \mathrm{cut}(\mathscr{A} \otimes_i \mathscr{B} \mid \mathscr{P} \otimes_j \mathscr{Q}) \vdash \phi[+i] : \Phi, \psi[+i] : \Psi, \varphi[+j] : \Upsilon}$$

  is transformed into the derivation tree

$$\text{(Cut)}\ \dfrac{\overset{v}{\gamma : \Gamma \dashv \Sigma \vdash \mathscr{B} : B, \psi : \Psi} \qquad \text{(Cut)}\ \dfrac{\overset{u}{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi} \qquad \overset{w}{\theta : \Theta, \mathscr{P} : A, \mathscr{Q} : B \dashv \Xi \vdash \varphi : \Upsilon}}{\delta : \Delta, \theta : \Theta, \mathscr{Q} : B \dashv \Pi \uplus \Xi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \vdash \phi : \Phi, \varphi : \Upsilon}}{\delta : \Delta, \gamma : \Gamma, \theta : \Theta \dashv \Pi \uplus \Sigma \uplus \Xi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \mathrm{cut}(\mathscr{B} \mid \mathscr{Q}) \vdash \phi : \Phi, \psi : \Psi, \varphi : \Upsilon}$$

- ($\parr$R, $\parr$L)-Cut. A derivation tree of the form

$$\text{(Cut)}\ \dfrac{\text{($\parr$R)}\ \dfrac{\overset{u}{\theta : \Theta \dashv \Xi \vdash \mathscr{A} : A, \mathscr{B} : B, \varphi : \Upsilon}}{\theta[+i] : \Theta \dashv \Xi[+i] \vdash \mathscr{A} \parr_i \mathscr{B} : A \parr B, \varphi[+i] : \Upsilon} \qquad \text{($\parr$L)}\ \dfrac{\overset{v}{\delta : \Delta, \mathscr{P} : A \dashv \Pi \vdash \phi : \Phi} \qquad \overset{w}{\gamma : \Gamma, \mathscr{Q} : B \dashv \Sigma \vdash \psi : \Psi}}{\delta[+j] : \Delta, \gamma[+j] : \Gamma, \mathscr{P} \parr_j \mathscr{Q} : A \parr B \dashv \Pi[+j], \Sigma[+j] \vdash \phi[+j] : \Phi, \psi[+j] : \Psi}}{\delta[+j] : \Delta, \gamma[+j] : \Gamma, \theta[+i] : \Theta \dashv \Pi[+j] \uplus \Sigma[+j] \uplus \Xi[+i], \mathrm{cut}(\mathscr{A} \parr_i \mathscr{B} \mid \mathscr{P} \parr_j \mathscr{Q}) \vdash \phi[+j] : \Phi, \psi[+j] : \Psi, \varphi[+i] : \Upsilon}$$

is transformed into the derivation tree

$$\text{(CUT)}\ \dfrac{\text{(}\parr\text{L)}\ \dfrac{\overset{u}{\theta : \Theta \dashv \Xi \vdash \mathscr{A} : A, \mathscr{B} : B, \varphi : \Upsilon} \qquad \overset{v}{\delta : \Delta, \mathscr{P} : A \dashv \Pi \vdash \phi : \Phi}}{\delta : \Delta, \theta : \Theta \dashv \Pi \uplus \Xi \uplus \mathrm{cut}(\mathscr{A}, \mathscr{P}) \vdash \mathscr{B} : B, \phi : \Phi, \varphi : \Upsilon} \qquad \overset{w}{\gamma : \Gamma, \mathscr{Q} : B \dashv \Sigma \vdash \psi : \Psi}}{\delta : \Delta, \gamma : \Gamma, \theta : \Theta \dashv \Pi \uplus \Sigma \uplus \Xi \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \uplus \mathrm{cut}(\mathscr{B} \mid \mathscr{Q}) \vdash \phi : \Phi, \psi : \Psi, \varphi : \Upsilon}$$

- (&R, &L)-Cut. A derivation tree of the form

$$\text{(CUT)}\ \dfrac{\text{(\&R)}\ \dfrac{\overset{u_1}{\delta_1 : \Delta \dashv \Pi_1 \vdash \mathscr{A} : A, \phi_1 : \Phi} \qquad \overset{u_2}{\delta_2 : \Delta \dashv \Pi_2 \vdash \mathscr{B} : B, \phi_2 : \Phi}}{\delta_1[+i], \delta_2[+j] : \Delta \dashv \Pi_1[+i] \uplus \Pi_2[+j] \vdash \mathscr{A} \&_i, \&_j \mathscr{B} : A \,\&\, B, \phi_1[+i], \phi_2[+j] : \Phi} \qquad \text{(}\otimes\text{L)}\ \dfrac{\overset{v}{\gamma : \Gamma, \mathscr{P} : A \dashv \Sigma \vdash \psi : \Psi}}{\gamma[+k] : \Gamma, \pi^{\&}_{1,k}(\mathscr{P}) : A \,\&\, B \dashv \Sigma[+k] \vdash \psi[+k] : \Psi}}{\delta_1[+i], \delta_2[+j] : \Delta, \gamma[+k] : \Gamma \dashv \Pi_1[+i] \uplus \Pi_2[+j] \uplus \Sigma[+k] \uplus \mathrm{cut}(\mathscr{A} \&_i, \&_j \mathscr{B} \mid \pi^{\&}_{1,k}(\mathscr{P})) \vdash \phi_1[+i], \phi_2[+j] : \Phi, \psi : \Psi}$$

is transformed into the derivation tree

$$\text{(CUT)}\ \dfrac{\overset{u_1}{\delta_1 : \Delta \dashv \Pi_1 \vdash \mathscr{A} : A, \phi_1 : \Phi} \qquad \overset{v}{\gamma : \Sigma, \mathscr{P} : A \dashv \Sigma \vdash \psi : \Psi}}{\delta_1 : \Delta, \gamma : \Gamma \dashv \Pi_1 \uplus \Sigma \uplus \mathrm{cut}(\mathscr{A} \mid \mathscr{P}) \vdash \phi_1 : \Phi, \psi : \Psi}$$

and a derivation tree of the form

$$\text{(CUT)}\ \dfrac{\text{(\&R)}\ \dfrac{\overset{u_1}{\delta_1 : \Delta \dashv \Pi_1 \vdash \mathscr{A} : A, \phi_1 : \Phi} \qquad \overset{u_2}{\delta_2 : \Delta \dashv \Pi_2 \vdash \mathscr{B} : B, \phi_2 : \Phi}}{\delta_1[+i], \delta_2[+j] : \Delta \dashv \Pi_1[+i] \uplus \Pi_2[+j] \vdash \mathscr{A} \&_i, \&_j \mathscr{B} : A \,\&\, B, \phi_1[+i], \phi_2[+j] : \Phi} \qquad \text{(}\otimes\text{L)}\ \dfrac{\overset{v}{\gamma : \Gamma, \mathscr{Q} : B \dashv \Sigma \vdash \psi : \Psi}}{\gamma[+k] : \Gamma, \pi^{\&}_{2,k}(\mathscr{Q}) : A \,\&\, B \dashv \Sigma[+k] \vdash \psi[+k] : \Psi}}{\delta_1[+i], \delta_2[+j] : \Delta, \gamma[+k] : \Gamma \dashv \Pi_1[+i] \uplus \Pi_2[+j] \uplus \Sigma[+k] \uplus \mathrm{cut}(\mathscr{A} \&_i, \&_j \mathscr{B} \mid \pi^{\&}_{2,k}(\mathscr{Q})) \vdash \phi_1[+i], \phi_2[+j] : \Phi, \psi : \Psi}$$

is transformed into the derivation tree

$$\text{(CUT)}\ \dfrac{\overset{u_2}{\delta_2 : \Delta \dashv \Pi_2 \vdash \mathscr{B} : B, \phi_2 : \Phi} \qquad \overset{v}{\gamma : \Gamma, \mathscr{Q} : B \dashv \Sigma \vdash \psi : \Psi}}{\delta_2 : \Delta, \gamma : \Gamma \dashv \Pi_2 \uplus \Sigma \uplus \mathrm{cut}(\mathscr{B} \mid \mathscr{Q}) \vdash \phi_2 : \Phi, \psi : \Psi}$$

- (⊕R, ⊕L)-Cut. A derivation tree of the form

$$\text{(CUT)}\ \dfrac{\text{(}\oplus\text{R)}\ \dfrac{\overset{u}{\gamma : \Gamma \dashv \Sigma \vdash \mathscr{A} : A, \psi : \Psi}}{\gamma[+i] : \Gamma \dashv \Sigma[+i] \vdash \iota^{\oplus}_{1,i}(\mathscr{A}) : A \oplus B, \psi[+i] : \Psi} \qquad \text{(}\oplus\text{L)}\ \dfrac{\overset{v_1}{\delta_1 : \Delta, \mathscr{P} : A \dashv \Pi_1 \vdash \phi_1 : \Phi} \qquad \overset{v_2}{\delta_2 : \Delta, \mathscr{Q} : B \dashv \Pi_2 \vdash \phi_2 : \Phi}}{\delta_1[+j], \delta_2[+k] : \Delta, \mathscr{P} \oplus_j, \oplus_k \mathscr{Q} : A \oplus B \dashv \Pi_1[+j] \uplus \Pi_2[+k] \vdash \phi_1[+j], \phi_2[+k] : \Phi}}{\delta_1[+j], \delta_2[+k] : \Delta, \gamma[+i] : \Gamma \dashv \Pi_1[+j] \uplus \Pi_2[+k] \uplus \Sigma[+i] \uplus \mathrm{cut}(\iota^{\oplus}_{1,i}(\mathscr{A}) \mid \mathscr{P} \oplus_j, \oplus_k \mathscr{Q}) \vdash \phi_1[+j], \phi_2[+k] : \Phi, \psi[+i] : \Psi}$$

is transformed into the derivation tree

$$\text{(CUT)}\ \dfrac{\overset{u}{\gamma : \Gamma \dashv \Sigma \vdash \mathscr{A} : A, \psi : \Psi} \qquad \overset{v_1}{\delta_1 : \Delta, \mathscr{P} : A \dashv \Pi_1 \vdash \phi_1 : \Phi}}{\delta_1 : \Delta, \gamma : \Gamma \dashv \Pi_1 \uplus \Sigma \uplus \mathrm{cut}(\mathscr{A}, \mathscr{P}) \vdash \phi_1 : \Phi, \psi : \Psi}$$

and a derivation tree of the form

$$\text{(CUT)}\ \dfrac{\text{(}\oplus\text{R)}\ \dfrac{\overset{u}{\gamma : \Gamma \dashv \Sigma \vdash \mathscr{B} : B, \psi : \Psi}}{\gamma[+i] : \Gamma \dashv \Sigma[+i] \vdash \iota^{\oplus}_{2,i}(\mathscr{B}) : A \oplus B, \psi[+i] : \Psi} \qquad \text{(}\oplus\text{L)}\ \dfrac{\overset{v_1}{\delta_1 : \Delta, \mathscr{P} : A \dashv \Pi_1 \vdash \phi_1 : \Phi} \qquad \overset{v_2}{\delta_2 : \Delta, \mathscr{Q} : B \dashv \Pi_2 \vdash \phi_2 : \Phi}}{\delta_1[+j], \delta_2[+k] : \Delta, \mathscr{P} \oplus_j, \oplus_k \mathscr{Q} : A \oplus B \dashv \Pi_1[+j] \uplus \Pi_2[+k] \vdash \phi_1[+j], \phi_2[+k] : \Phi}}{\delta_1[+j], \delta_2[+k] : \Delta, \gamma[+i] : \Gamma \dashv \Pi_1[+j] \uplus \Pi_2[+k] \uplus \Sigma[+i] \uplus \mathrm{cut}(\iota^{\oplus}_{2,i}(\mathscr{B}) \mid \mathscr{P} \oplus_j, \oplus_k \mathscr{Q}) \vdash \phi_1[+j], \phi_2[+k] : \Phi, \psi[+i] : \Psi}$$

is transformed into the derivation tree

$$\text{(CUT)}\ \dfrac{\overset{u}{\gamma : \Gamma \dashv \Sigma \vdash \mathscr{B} : B, \psi : \Psi} \qquad \overset{v_2}{\delta_2 : \Delta, \mathscr{Q} : B \dashv \Pi_2 \vdash \phi_2 : \Phi}}{\delta_2 : \Delta, \gamma : \Gamma \dashv \Pi_2 \uplus \Sigma \uplus \mathrm{cut}(\mathscr{B}, \mathscr{Q}) \vdash \phi_1 : \Phi, \psi : \Psi}$$

- (!R, !D)-Cut. A derivation tree of the form

$$\text{(CUT)}\ \dfrac{\text{(!R)}\ \dfrac{\overset{u}{\delta : !\Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : ?\Phi}}{\delta[+i] : !\Delta \dashv \Pi[+i] \vdash !_i \mathscr{A} : !A, \phi[+i] : ?\Phi} \qquad \text{(!D)}\ \dfrac{\overset{v}{\gamma : \Gamma, \mathscr{P} : A \dashv \Sigma \vdash \psi : \Psi}}{\gamma[+j] : \Gamma, !_j \mathscr{P} : !A \dashv \Sigma[+j] \vdash \psi[+j] : \Psi}}{\delta[+i] : !\Delta, \gamma[+j] : \Gamma \dashv \Pi[+i] \uplus \Sigma[+j] \uplus \mathrm{cut}(!_i \mathscr{A}, !_j \mathscr{P}) \vdash \phi[+i] : ?\Phi, \psi[+j] : \Psi}$$

is transformed into the derivation tree

$$(\textsc{Cut})\ \dfrac{\overset{u}{\delta : !\Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : ?\Phi} \qquad \overset{v}{\gamma : \Gamma, \mathscr{P} : A \dashv \Sigma \vdash \psi : \Psi}}{\delta : !\Delta, \gamma : \Gamma \dashv \Pi \uplus \Sigma \uplus \mathrm{cut}(\mathscr{A}, \mathscr{P}) \vdash \phi : ?\Phi, \psi : \Psi}$$

- (!R, !W)-Cut. A derivation tree of the form

$$(\textsc{Cut})\ \dfrac{(!\mathrm{R})\ \dfrac{\overset{u}{\delta : !\Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : ?\Phi}}{\delta[+i] : !\Delta \dashv \Pi[+i] \vdash !_i\mathscr{A} : !A, \phi[+i] : ?\Phi} \qquad (!\mathrm{W})\ \dfrac{\overset{v}{\gamma : \Gamma \dashv \Sigma \vdash \psi : \Psi}}{\gamma : \Gamma, \emptyset : !A \dashv \Sigma \vdash \psi : \Psi}}{\delta[+i] : !\Delta, \gamma : \Gamma \dashv \Pi[+i] \uplus \Sigma \uplus \mathrm{cut}(!_i\mathscr{A}, \emptyset) \vdash \phi[+i] : ?\Phi, \psi : \Psi}$$

is transformed into the derivation tree

$$\overset{u}{\gamma : \Gamma \dashv \Sigma \vdash \psi : \Psi}$$

- (!R, !C)-Cut. A derivation tree of the form

$$(\textsc{Cut})\ \dfrac{(!\mathrm{R})\ \dfrac{\overset{u}{\delta : !\Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : ?\Phi}}{\delta[+i] : !\Delta \dashv \Pi[+i] \vdash !_i\mathscr{A} : !A, \phi[+i] : ?\Phi} \qquad (!\mathrm{C})\ \dfrac{\overset{v}{\gamma : \Gamma, \mathscr{P} : !A, \rho[\mathcal{S}] : !A \dashv \Sigma \vdash \psi : \Psi}}{\gamma : \Gamma, \mathscr{P} \uplus \rho[\mathcal{S}] : !A \dashv \Sigma \vdash \psi : \Psi}}{\delta[+i] : !\Delta, \gamma : \Gamma \dashv \Pi[+i] \uplus \Sigma \uplus \mathrm{cut}(!_i\mathscr{A}, \mathscr{P} \uplus \rho[\mathcal{S}]) \vdash \phi[+i] : ?\Phi, \psi : \Psi}$$

is transformed into the derivation tree

$$(?\mathrm{C}^*)\ \dfrac{(!\mathrm{C}^*)\ \dfrac{(\textsc{Cut})\ \dfrac{(!\mathrm{R})\ \dfrac{\overset{u}{\delta : !\Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : ?\Phi}}{\delta[+j] : !\Delta \dashv \Pi[+j] \vdash !_j\mathscr{A} : !A, \phi[+j] : ?\Phi} \qquad (\textsc{Cut})\ \dfrac{(!\mathrm{R})\ \dfrac{\overset{v}{\delta : !\Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : ?\Phi}}{\delta[+i] : !\Delta \dashv \Pi[+i] \vdash !_i\mathscr{A} : !A, \phi[+i] : ?\Phi} \qquad \overset{v}{\gamma : \Gamma, \mathscr{P} : !A, \rho[\mathcal{S}] : !A \dashv \Sigma \vdash \psi : \Psi}}{\delta[+i] : !\Delta, \gamma : \Gamma, \rho[\mathcal{S}] : !A \dashv \Pi[+i] \uplus \Sigma \uplus \mathrm{cut}(!_i\mathscr{A} \mid \mathscr{P}) \vdash \phi[+i] : ?\Phi, \psi : \Psi}}{\delta[+j] : !\Delta, \delta[+i] : !\Delta, \gamma : \Gamma \dashv \Pi[+j] \uplus \Pi[+i] \uplus \Sigma \uplus \mathrm{cut}(!_j\mathscr{A} \mid \rho[\mathcal{S}]) \uplus \mathrm{cut}(!_i\mathscr{A} \mid \mathscr{P}) \vdash \phi[+j] : ?\Phi, \phi[+i] : ?\Phi, \psi : \Psi}}{\delta[+j] \uplus \delta[+i] : !\Delta, \gamma : \Gamma \dashv \Pi[+j] \uplus \Pi[+i] \uplus \Sigma \uplus \mathrm{cut}(!_j\mathscr{A} \mid \rho[\mathcal{S}]) \uplus \mathrm{cut}(!_i\mathscr{A} \mid \mathscr{P}) \vdash \phi[+j] : ?\Phi, \phi[+i] : ?\Phi, \psi : \Psi}}{\delta[+j] \uplus \delta[+i] : !\Delta, \gamma : \Gamma \dashv \Pi[+j] \uplus \Pi[+i] \uplus \Sigma \uplus \mathrm{cut}(!_j\mathscr{A} \mid \rho[\mathcal{S}]) \uplus \mathrm{cut}(!_i\mathscr{A} \mid \mathscr{P}) \vdash \phi[+j] \uplus \phi[+i] : ?\Phi, \psi : \Psi}$$

- (?D, ?L)-Cut. A derivation tree of the form

$$(\textsc{Cut})\ \dfrac{(?\mathrm{D})\ \dfrac{\overset{u}{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi}}{\delta[+i] : \Delta \dashv \Pi[+i] \vdash ?_i\mathscr{A} : ?A, \phi[+i] : \Phi} \qquad (?\mathrm{L})\ \dfrac{\overset{v}{\gamma : !\Gamma, \mathscr{P} : A \dashv \Sigma \vdash \psi : ?\Psi}}{\gamma[+j] : !\Gamma, ?_j\mathscr{P} : ?A \dashv \Sigma[+j] \vdash \psi[+j] : ?\Psi}}{\delta[+i] : \Delta, \gamma[+j] : !\Gamma \dashv \Pi[+i] \uplus \Sigma[+j] \uplus \mathrm{cut}(?_i\mathscr{A}, ?_j\mathscr{P}) \vdash \phi[+i] : \Phi, \psi[+j] : ?\Psi}$$

is transformed into the derivation tree

$$(\textsc{Cut})\ \dfrac{\overset{u}{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi} \qquad \overset{v}{\gamma : !\Gamma, \mathscr{P} : A \dashv \Sigma \vdash \psi : ?\Psi}}{\delta : \Delta, \gamma : !\Gamma \dashv \Pi \uplus \Sigma \uplus \mathrm{cut}(\mathscr{A}, \mathscr{P}) \vdash \phi : \Phi, \psi : ?\Psi}$$

- (?W, ?L)-Cut. This case is impossible due to the quantity restriction on the rule Cut.
- (?C, ?L)-Cut. This case is impossible due to the quantity restriction on the rule Cut.

A.2 Right minor cuts

The second pattern is an application of Cut whose principal formula at the end of the right hypothesis of the application of Cut is not the cut formula. Unlike cut-elimination for sequent calculi, however, this pattern is irrelevant to the $\ell\beta$-reduction, so we skip it.

A.3 Left minor cuts

The third pattern is an application of the rule Cut whose principal formula at the end of the left hypothesis of the application of Cut is not the cut formula. Again, this patter is irrelevant to the $\ell\beta$-reduction, so we skip it.

A.4 Identity cuts

Finally, the fourth pattern is an application of the rule Cut such that at least one of the two hypotheses is a singleton derivation of the axiom Ax.

- Left Ax-Cut. A derivation tree of the form

$$\text{(Cut)}\ \dfrac{\text{(Ax)}\ \dfrac{}{\underline{x}_i[j] : A \dashv\vdash \overline{x}_j[i] : A} \qquad \overset{u}{\delta : \Delta, \mathscr{P} : A \dashv \Pi \vdash \phi : \Phi}}{\delta : \Delta, \underline{x}_i[j] : A \dashv \Pi \uplus \mathrm{cut}(\overline{x}_j[i], \mathscr{P}) \vdash \phi : \Phi}$$

is transformed into the derivation tree

$$\overset{u}{\delta : \Delta, \mathscr{P} : A \dashv \Pi \vdash \phi : \Phi}$$

- Right Ax-Cut. A derivation tree of the form

$$\text{(Cut)}\ \dfrac{\overset{u}{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi} \qquad \text{(Ax)}\ \dfrac{}{\underline{x}_i[j] : A \dashv\vdash \overline{x}_j[i] : A}}{\delta : \Delta \dashv \Pi, \mathrm{cut}(\mathscr{A}, \underline{x}_i[j]) \vdash \overline{x}_j[i] : A, \phi : \Phi}$$

is transformed into the derivation tree

$$\overset{u}{\delta : \Delta \dashv \Pi \vdash \mathscr{A} : A, \phi : \Phi}$$