# Game Semantics for the Consistency of Martin-Löf Type Theory with Church's Thesis

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# Table of Contents

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CT is false in recursion theory, but expected to be consistent with MLTT because objects in MLTT are all *constructive*.

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2. Realizability *à la assemblies* models MLTT, but not CT.

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$$(\xi)\ \frac{x : \mathsf{N} \vdash x = x + 0 : \mathsf{N}}{\vdash \lambda x^{\mathsf{N}}.\, x = \lambda x^{\mathsf{N}}.\, x + 0 : \mathsf{N} \Rightarrow \mathsf{N}}$$

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**Problem (in a nutshell)**

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**Problem (slightly expositive)**

*Realizability can only be either intensional or extensional, but the consistency problem requires it to be both. Impossible!*

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**Remark (Why the idea would work)**

- For CT, the calculation of realizers in this game semantics would be 'copy-cat,' like Kleene, trivially validating CT.
- It would be as extensional as the existing game semantics (n.b., realizers not assigned to mor.), validating $\xi$ (and modeling MLTT).

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- Models *effects* systematically;
- New directions: I have applied game semantics to recursion theory, constructive math, category theory, categorical logic, etc.

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## Remark (Not quite the conventional game semantics)

The present work enlarges the asymmetry between P and O in standard games, and so it is novel from the game-semantic point of view as well.

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Because the main part of CT has nothing to do with dependent types, we work on [Abramsky and McCusker, 1999] in this talk.

I.e., we focus on how to model the term $\mathsf{ct} : (\mathsf{N} \Rightarrow \mathsf{N}) \Rightarrow \mathsf{N}$ that outputs a realizer $\mathsf{e} : \mathsf{N}$ for a given input $\mathsf{f} : \mathsf{N} \Rightarrow \mathsf{N}$.

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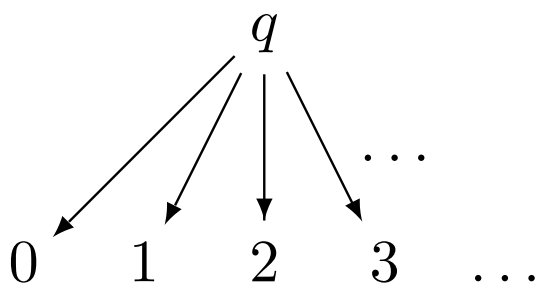
**Example (Games)**

The game $N$ of natural numbers and the game $\mathbf{0}$ of falsity are

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**Remark (Linear implication)**

Strictly speaking, the construction $\Rightarrow$ models *linear implication*. However, this simplification is not harmful to the rest of the talk.

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## Remark

O plays the role of P in the domain $A$ of each implication $A \Rightarrow B$.

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A ***strategy*** $\sigma$ on a game $G$, written $\sigma : G$, is a partial map

$$\sigma : \text{odd-length position } m_1 m_2 \dots m_{2i+1} \text{ in } G \mapsto \text{P-move } m_{2i+2} \text{ in } G$$

s.t. the concatenation $m_1 m_2 \dots m_{2i+1} m_{2i+2}$ is a position in $G$.

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**Theorem (Consistency of MLTT + CT)**

*MLTT is consistent with CT.*

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## Further work

1. Thanks to the *compositionality* of my method, one would easily apply the present method to extensions of MLTT + CT.
2. Generally, there would be more meta-theoretic problems on MLTT for which game semantics works well.


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