Internal ∞-groupoids and computational game semantics of homotopy type theory

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Abstract

We present game semantics of homotopy type theory (HoTT) that interprets terms of HoTT by *computations*. HoTT is to serve as a computational foundation of mathematics, but existing semantics of HoTT has been mostly *homotopical*. This bottlenecks the development of HoTT as a computational foundation of mathematics. For this reason, *computational* semantics of HoTT such as our game semantics has been sought ardently for a long time. Our game semantics is obtained from Warren's strict ∞-groupoid interpretation of type theory by weakening the invertibility of nonzero-cells in such a way that it validates the univalence axiom, the heart of HoTT, and internalising it to the game semantics of type theory. For generality and lucidness, we first define our ∞-groupoid interpretation internal to categorical semantics of type theory, called *categories with families*, and only later focus on its game-semantic instance. Hence, most part of this article is on category theory; only the last part is on game semantics. Notably, our internalisation does not require even all pullbacks in categories with families so that our method is applicable to a wide range of concrete semantics. This point is crucial for the present work since the category with families of games does not have all pullbacks. We employ game semantics, among various computational semantics, because its distinguished *intensionality* is useful for the study of HoTT, e.g., it proves the *independence of Markov's principle* from HoTT.

Contents

1 Introduction

1.1 Background

Since the work in the 19–20th centuries by Frege, Peano, Russell, Hilbert, Gentzen and others, rigorous formalisations of mathematics by symbolic means, called *formal systems* [TS00], have been advanced.

Intensional Martin-Löf type theory (MLTT)[1] [ML75, ML84, ML98] is one of the best-known formal systems for *constructive mathematics* [TvD88], i.e., computational or *constructive* part of mathematics. MLTT is comparable to axiomatic set theory [Zer08, Fra22] for classical mathematics. For example, just like axiomatic set theory is explained or justified by *sets* in an informal sense, the conceptual foundation of MLTT is *computations* in an informal sense. In other words, the fundamental idea of MLTT is

> To regard objects (e.g., sets and functions) and proofs uniformly as computations,

where, in contrast with the separation of logic from set theory, MLTT regards proofs as a particular class of objects. Hence, objects and proofs in MLTT are unified into *terms*, while formulas in MLTT are called *types*. This standard yet informal interpretation of MLTT is called the *Brouwer-Heyting-Kolmogorov (BHK) interpretation*[2] [Bro23, Hey30, Hey31, Hey34, Kol32] and made precise by mathematical formalisations of computations such as *realisability* [Kle45, Str12]. On the other hand, being a symbolic formalisation of the BHK-interpretation, it is no surprise that MLTT is also a programming language [ML82, NPS90].

However, MLTT is hardly sufficient for *higher-dimensional* structures and reasonings on *equality*. This situation changed with the advent of *homotopy type theory (HoTT)* [Uni13, §A.2], a breakthrough extension of MLTT by *higher inductive types (HITs)* and Voevodsky's beautiful *univalence axiom (UA)*. HITs are inductive constructions on higher-dimensional objects that assist a formalisation of homotopy theory. They give us a new, *synthetic* method to formalise homotopy theory, which is often simpler and more transparent than the traditional approach. On the other hand, UA formally identifies isomorphic objects and makes mathematics invariant between isomorphic objects. This practice is very common in mathematics yet quite tedious to make precise, so its formalisation is a mathematician's dream.

The development of HoTT has been based on the *homotopical* semantics of MLTT [KL12, AW09], which interprets formulas as *(topological) spaces*, proofs as *points* in spaces, and identity proofs as *paths* or *homotopies*. Based on this homotopical view, however, HoTT has lost the BHK-interpretation:

> The homotopical semantics no longer sees proofs or objects as computations, and it has been unknown how to apply the BHK-interpretation to HoTT.

For this reason, it has been unclear how to make sense of HoTT as a formalisation of computations. This fundamental problem bottlenecks the development of HoTT as a foundation of constructive mathematics or as a programming language. Accordingly, it has been a central problem to establish a *computational* interpretation of HoTT [Uni13, p. 16].

Even from the viewpoint of classical mathematics, the homotopical interpretation is arguable for HoTT to serve as a foundation of mathematics because it does not make much sense for the logical part of HoTT. Specifically, the problem is:

[1] The other, *extensional* Martin-Löf type theory, which is the intensional one augmented with the law of *equality reflection* or *I-elimination* [ML82, p. 169], cannot be extended to homotopy type theory [Uni13, Example 3.1.9]. For this reason, we focus on the intensional one and call it MLTT in the present article.

[2] Strictly speaking, the BHK-interpretation is only about constructive logic, and it is extended to entire constructive mathematics by Martin-Löf's *meaning explanation* for MLTT [ML84]. Nevertheless, for simplicity, we do not distinguish the two terminologies in the present article.

> Intuitively, proofs are reasonings on the truths of formulas, not points in spaces. Similarly, identity proofs are reasonings on the equality between terms, not paths in spaces.

In contrast, the BHK-interpretation interprets proofs as such (computational) reasonings, so it justifies MLTT as a foundation of mathematics in a highly convincing fashion. Hence, its extension to HoTT is sought for because it will make HoTT highly convincing as a foundation of mathematics.

1.2 Main results

Motivated in this way, we achieve such an extension mathematically in the style of *game semantics*:

Theorem 1.1 (game semantics of HoTT)**.** *There is a mathematically precise formulation of the BHK-interpretation of HoTT in the style of game semantics, which interprets terms in HoTT as interactive computations between an algorithm (or prover) and an oracle (or refuter) in games.*

Specifically, HoTT in this theorem refers to MLTT equipped with One-, Zero-, N-, Pi-, Sigma- and Id-types, and univalent universes. Due to space limitation, we leave it as future work to interpret HITs.

Game semantics [Hyl97, A$^+$97] is a class of *mathematical semantics* [Gun92, AC$^+$98] of formal systems, which interprets formulas or types by *games* between a *prover* and a *refuter*, and proofs/objects or terms by *strategies* for the prover to *win* games. Technically, games are a class of rooted directed trees, and strategies are algorithms for the prover on *how to walk* on games alternately with the refuter in such a way that it leads to a *win* for the prover. A strong point of game semantics is its *conceptual naturality*: Mathematics can be seen as games between the prover and the refuter, and game semantics is nothing but what makes this intuitive idea precise. Another advantage is its *intensionality*: Game semantics is unique in its interpretation of terms by strategies or *intensional processes*, while other semantics interprets terms by extensional objects such as functions. Because terms are also intensional, computing in a step-by-step fashion, game semantics achieves a very tight correspondence between terms and strategies, which makes game semantics an exceptionally powerful tool for the study of formal systems.

We choose game semantics as our mathematical framework to formalise the BHK-interpretation, among other computational semantics, for these distinguished advantages. See §1.4–1.5 on this point.

Theorem 1.1 solves the central problem articulated in §1.1 by formulating the BHK-interpretation of HoTT in a mathematically precise yet intuitive fashion. In other words, the theorem establishes a firm mathematical ground for analysing, justifying and developing HoTT as a foundation of constructive mathematics and as a programming language, subsuming the constructivity of the univalence axiom.

Besides, the theorem is a mathematical breakthrough because game semantics is *orthogonal* to the currently dominant, homotopical interpretation. In particular:

> Our game semantics interprets identity proofs between two terms as computations that witness the equality between the corresponding two computations,

which is convincing as a computational interpretation of identity proofs and in line with the BHK-interpretation. For instance, the game semantics interprets an identity proof between function terms $f, g : A \to B$ as computation verifying that f and g have the same input-output behaviour. This interpretation is very different from the homotopical one which interprets identity proofs by paths/homotopies.

In addition to the novel computational interpretation, our game semantics is also the first *intensional* interpretation of HoTT. This intensionality is highly nontrivial since the univalence axiom is a strongly *extensional* principle. For instance, the univalence axiom implies *function extensionality* [Uni13, §4.9]. Moreover, the intensionality makes the game semantics effective for the study of HoTT, e.g.,

Corollary 1.2 (independence of Markov's principle from HoTT)**.** *The game semantics of HoTT refutes Markov's principle [Mar62], thus proving the independence of the principle from HoTT.*

Markov's principle (in arithmetic form [TvD88, p. 204]) states that if it is not impossible for a function $f : \mathbb{N} \to \mathbb{N}$ to output 0, then there is an input $n \in \mathbb{N}$ such that $f(n) = 0$:

$$\forall f : \mathbb{N} \to \mathbb{N}. \neg\neg(\exists n \in \mathbb{N}. f(n) = 0) \Rightarrow \exists n \in \mathbb{N}. f(n) = 0. \quad (1)$$

Markov proposed (1) as a constructively valid one by describing an informal algorithm to validate it (though it forms an instance of the law of *double negation elimination*, which is in general constructively invalid). However, other constructivists, including Brouwer, objected this proposal [TvD88, p. 27]. In fact, it is a delicate matter to decide if Markov's principle is constructive, and it is unsettled even today [TvD88, §4.5.1]. Notably, standard computational semantics of type theory such as Hyland's effective topos [Hyl82] validates Markov's principle. Hence, the corollary demonstrates the power of our game semantics (though the sheaf semantics of cubical type theory [CRS21] also refutes Markov's principle).

Finally, our game semantics does not only validates the univalence axiom but also its stronger, *judgemental* variant, i.e., equivalence is judgementally equal to equality. Consequently:

Corollary 1.3 (consistency of judgemental univalence)**.** *The judgemental univalence axiom is consistent with MLTT.*

Although the PhD thesis [Win20, p. 63] of Winterhalter also proves this consistency result, our game semantics is, to the best of our knowledge, the first computational semantics that proves the consistency.

1.3 Sketch of our approach

Our approach is based on Warren's (strict) ∞-groupoid interpretation of MLTT [War11]. Specifically, our game semantics of HoTT is obtained from his ∞-groupoid interpretation by first weakening the invertibility of nonzero-cells and then internalising it to the author's game semantics of MLTT [Yam19]. We note that, although there are other variants of game semantics of MLTT [AJV15, BL18], only the chosen one interprets all the types necessary for the present work, including the hierarchy of universes. The weakening of the invertibility is our main idea to interpret the univalence axiom constructively.

For generality and lucidness, we do not directly construct the game semantics of HoTT but first construct the weakened ∞-groupoid interpretation internal to a wide range of semantics of MLTT and only later focus on its game-semantic instance. To this end, we adopt Dybjer's *categories with families (CwFs)* [Dyb96], one of the most established categorical semantics of MLTT. Specifically, we first construct a CwF of ∞-groupoids (in which inverses are weak) internal to a given CwF and interpret HoTT by this CwF. We then specialise in the CwF of ∞-groupoids inetrnal to the CwF of games.

Notably, this internalisation of ∞-groupoids does not require all pullbacks in the CwFs. In this sense, our technique is more general than the standard internalisation of categories via pullbacks [Bor94, §8]. This generality is crucial because the CwF of games does not have all pullbacks.

Because we instantiate our categorical interpretation of HoTT by games only in the last section (§5), the present work is accessible to various audiences and applicable to a wide range of semantics. The reader only interested in the categorical semantics can skip the last section and focus on the other sections.

1.4 Comparison with realisability

One may wonder if other computational semantics of MLTT such as realisability [Kle45] can be extended to HoTT by the same technique. The answer seems affirmative, i.e., we can obtain realisability semantics of HoTT by internalising the weakened ∞-groupoid interpretation to the CwF of realisability [Reu99, Str08, Str12] except that we have to generalise realisable (total) functions to *partial* ones for the interpretation of universes (though we have not checked all the details).

Nevertheless, the present work employs game semantics, not realisability, for the following reasons. First, by the intensionality of strategies, game semantics verifies the independence of Makov's principle from HoTT (§1.2), while realisability does not due to the extensionality of realisable functions [Hyl82]. This demonstrates an advantage of game semantics over realisability for the study of HoTT.

Next, it is mathematically more nontrivial to achieve game semantics of HoTT than realisability. This is because HoTT postulates the univalence axiom, which is strongly *extensional*, while game semantics is more intensional than realisability. From another angle, realisability seems to be a *corollary* of game semantics: We obtain the former by *extensionally collapsing* the latter, i.e., by collapsing strategies $\phi : A \to B$ into functions $\alpha : A \mapsto \phi \circ \alpha : B$ (though we have not checked all the details).

Finally, again by its intensionality, game semantics fits better to our weakened invertibility of nonzero-cells than realisability. For instance, we can simply internalise Warren's ∞-groupoids in realisability, in which inverses are *strict*. In contrast, for game semantics, we must weaken inverses since, e.g., even the game semantics of the implication $\mathsf{N} \Rightarrow \mathsf{N}$ from the natural number type N to itself contains strategies that do not have strict inverses. Thus, game semantics is a *representative* or *native* of our method, and therefore it explains the intuition behind the categorical semantics of HoTT better than realisability.

1.5 Our contributions and related work

Our main contribution is the BHK-interpretation of HoTT in the style of game semantics. Its significance is its *computational* nature since it is orthogonal to the currently dominant, homotopical interpretation. Another significance is its *intensionality* since HoTT postulates the univalence axiom, which is strongly extensional. By these novel features, our game semantics gives HoTT new insights and techniques.

For example, by interpreting HoTT in terms of the computational dialogical arguments between the prover and the refuter in games, we establish a firm ground for HoTT to serve as a foundation of constructive mathematics both conceptually and mathematically. In particular, this game semantics interprets the univalence axiom by computations, solving the central open problem in the field. Besides, our game semantics proves the independence of Markov's principle from HoTT.

Our main technical contribution is the construction of the CwF of ∞-groupoids (with weak inverses) internal to a given CwF. This construction is technically nontrivial because it is far from obvious that one may weaken the invertibility of Warren's ∞-groupoid interpretation. Moreover, the CwF of games does *not* have all pullbacks, so the standard method of internalising categories [Bor94, §8] does not work. We overcome this problem by constructing a certain class of pullbacks necessary for the internalisation of ∞-groupoids out of the structure of CwFs.

An existing computational interpretation of HoTT is given by the *cubical set model* [BCH14]. This model is essentially a computational variant of Voevodsky's *simplicial set model* [KL12]. Therefore, it is inherently homotopical, and its constructivity refers to that of the meta-theory. Also, the homotopical semantics interprets terms by points in spaces, and identity proofs by paths and homotopies, which does not make sense as an interpretation of proofs (§1.1). In contrast, our game semantics, being a variant of the BHK-interpretation, interprets terms *directly by computations*, which makes sense for proofs too.

Yet another computational approach to HoTT is *cubical type theory (CTT)* [CCHM18, OP16, AH$^+$18b, CHM18, CH19], a variant of HoTT based on the cubical set model. In CTT, *path types*, which represent the spaces of paths, play a central role instead of Id-types. CTT is *computational* in the sense that it is a functional programming language just like MLTT, and remarkably the univalence axiom is *derivable* as a theorem in CTT, providing a constructive justification of the univalence axiom. Unlike the present work, however, this result is not a constructive justification of HoTT in its original form [Uni13, §A.2].

Finally, Angiuli and Harper [AH18a] have proposed a generalisation of the BHK-interpretation for justifying CTT as a foundation of constructive mathematics. They generalise computations in the BHK-interpretation into *higher-dimensional* ones. Intuitively, however, there are no *dimensions* in proofs or reasonings about the truths of formulas. In this sense, this approach is somewhat mysterious. In contrast, our game semantics can be seen as a mathematical formulation of actual reasonings in mathematics.

1.6 Structure of the present article

The rest of this article is structured as follows. We first recall CwFs in §2. We then define a higher-dimensional CwF of ∞-groupoids with weak inverses internal to an arbitrary CwF in §3 and interpret HoTT by the higher-dimensional CwF in §4. Finally, we specialise in the interpretation of HoTT by the higher-dimensional CwF of ∞-groupoids internal to the CwF of games in §4 and as corollaries prove the consistency of the judgemental univalence axiom and the independence of Markov's principle in §6.

2 Categories with families

In this section we recall Dybjer's *categories with families (CwF)* [Dyb96], one of the most established categorical semantics of MLTT, because our interpretation of HoTT is described by this categorical framework. We leave the syntax of MLTT and its interpretation in CwFs to a standard article [Hof97].

Definition 2.1 (categories with families [Dyb96, Hof97])**.** A *category with families (CwF)* is a tuple $\mathcal{M} = (\mathcal{M}, \mathrm{Ty}, \mathrm{Tm}, \_\{\_\}, T, \_.\_, \mathrm{p}, \mathrm{v}, \langle\_,\_\rangle\_)$, where

- $\mathcal{M}$ is a category with a chosen terminal object $T \in \mathcal{M}$;
- Ty is an assignment of a set $\mathrm{Ty}(\Gamma)$ of *types* over Γ to each object $\Gamma \in \mathcal{M}$;
- Tm is an assignment of a set $\mathrm{Tm}(\Gamma, A)$ of *terms* of type A over Γ to each pair (Γ, A) of an object $\Gamma \in \mathcal{M}$ and a type $A \in \mathrm{Ty}(\Gamma)$;

- $\_\{\_\}$ is an assignment of a function $\_\{\phi\} : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Delta)$, called the *substitution on types*, and a family $(\_\{\phi\}_A)_{A \in \mathrm{Ty}(\Gamma)}$ of functions $\_\{\phi\}_A : \mathrm{Tm}(\Gamma, A) \to \mathrm{Tm}(\Delta, A\{\phi\})$, called the *substitutions on terms*, to each morphism $\phi : \Delta \to \Gamma$;

- $\_.\_$ is an assignment of an object $\Gamma.A \in \mathcal{M}$, called the *comprehension* of A, to each pair (Γ, A) of an object $\Gamma \in \mathcal{M}$ and a type $A \in \mathrm{Ty}(\Gamma)$;

- p and v are assignments of a morphism $\mathrm{p}_A : \Gamma.A \to \Gamma$ and a term $\mathrm{v}_A \in \mathrm{Tm}(\Gamma.A, A\{\mathrm{p}_A\})$, called the *first projection* and the *second projection* on A, respectively, to each pair (Γ, A) of an object $\Gamma \in \mathcal{M}$ and a type $A \in \mathrm{Ty}(\Gamma)$;

- $\langle \_, \_ \rangle_\_$ is an assignment of a morphism $\langle \phi, \check{\alpha} \rangle_A : \Delta \to \Gamma.A$, called the *extension* of ϕ by $\check{\alpha}$, to each triple $(\phi, A, \check{\alpha})$ of a morphism $\phi : \Delta \to \Gamma$, a type $A \in \mathrm{Ty}(\Gamma)$ and a term $\check{\alpha} \in \mathrm{Tm}(\Delta, A\{\phi\})$,

that satisfies, for each object $\Theta \in \mathcal{M}$, morphism $\varphi : \Theta \to \Delta$ and term $\alpha \in \mathrm{Tm}(\Gamma, A)$, the equations

$$A\{\mathrm{id}_\Gamma\} = A \qquad A\{\phi \circ \varphi\} = A\{\phi\}\{\varphi\} \qquad \alpha\{\mathrm{id}_\Gamma\}_A = \alpha \qquad \alpha\{\phi \circ \varphi\}_A = \alpha\{\phi\}_A\{\varphi\}_{A\{\phi\}}$$

$$\mathrm{p}_A \circ \langle \phi, \check{\alpha} \rangle_A = \phi \qquad \mathrm{v}_A\{\langle \phi, \check{\alpha} \rangle_A\} = \check{\alpha} \qquad \langle \phi, \check{\alpha} \rangle_A \circ \varphi = \langle \phi \circ \varphi, \check{\alpha}\{\varphi\}_{A\{\phi\}} \rangle_A \qquad \langle \mathrm{p}_A, \mathrm{v}_A \rangle_A = \mathrm{id}_{\Gamma.A}.$$

Notation. Given a morphism $\Phi : \Delta \to \Gamma$ and a type $A \in \mathrm{Ty}(\Gamma)$ in a CwF $\mathcal{M}$, we define a morphism $\phi^+_A := \langle \phi \circ \mathrm{p}_{A\{\phi\}}, \mathrm{v}_{A\{\phi\}} \rangle_A : \Delta.A\{\phi\} \to \Gamma.A$.

Definition 2.2 (contextuality [CD14])**.** A CwF $\mathcal{M}$ is *contextual* if there is a function $\mathrm{lth} : \mathcal{M}_0 \to \mathbb{N}$ that satisfies, for each object $\Delta \in \mathcal{M}$, $\mathrm{lth}(\Gamma) = 0$ if and only if $\Delta = T$, and $\mathrm{lth}(\Gamma) = n + 1$ if and only if there are unique object $\Gamma \in \mathcal{M}$ and type $A \in \mathrm{Ty}(\Gamma)$ such that $\mathrm{lth}(\Gamma) = n$ and $\Delta = \Gamma.A$.

Definition 2.3 (democracy [CD14])**.** A CwF $\mathcal{M}$ is *democratic* if each object $\Gamma \in \mathcal{M}$ is assigned a type $A_\Gamma \in \mathrm{Ty}(T)$ and an isomorphism $\iota_\Gamma : \Gamma \xrightarrow{\sim} T.A_\Gamma$.

For our *computational* semantics of HoTT, we make a *choice* on the democracy of $\mathcal{M}$, i.e., the type A_Γ and the isomorphism ι_Γ for each object $\Gamma \in \mathcal{M}$. We then write $\mathrm{d} : \mathcal{M}_0 \to \mathrm{Ty}(T)$ for the function $\Gamma \in \mathcal{M} \mapsto \mathrm{d}(\Gamma) := A_\Gamma \in \mathrm{Ty}(\Gamma)$. We omit the subscript $(\_)_\Gamma$ on ι when it does not bring confusion.

We shall focus on the following class of CwFs:

Definition 2.4 (standardness)**.** A CwF is *standard* if it is contextual and democratic.

Strictly speaking, CwFs only have the structures common to all types in MLTT. Hence, we need to equip them with additional structures, called *semantic type formers* [Hof97, §3.3], to interpret individual types. In the rest of the present section, we recall the semantic type formers for One-, Zero, Pi-, Sigma-, Id- and N-types as well as (not necessarily univalent) universes. A CwF equipped with these semantic type formers interpret MLTT, subsuming these standard types; see Hofmann [Hof97] for the details.

Definition 2.5 (semantic One-type [Hof97])**.** A CwF $\mathcal{M}$ *supports One-type* if

- (One-Form) There is a type $1^{[\Gamma]} \in \mathrm{Ty}(\Gamma)$ for each object $\Gamma \in \mathcal{M}$;
- (One-Intro) There is a term $o_\Gamma \in \mathrm{Tm}(\Gamma, 1^{[\Gamma]})$;
- (One-Elim) Given a type $A \in \mathrm{Ty}(\Gamma.1^{[\Gamma]})$ and terms $\alpha \in \mathrm{Tm}(\Gamma, A\{\overline{o_\Gamma}\})$ and $\tau \in \mathrm{Tm}(\Gamma, 1^{[\Gamma]})$, there is a term $\mathcal{R}^1_A(\alpha, \tau) \in \mathrm{Tm}(\Gamma, A\{\overline{\tau}\})$;
- (One-Comp) $\mathcal{R}^1_A(\alpha, o_\Gamma) = \alpha$;
- (One-Subst) $1^{[\Gamma]}\{\phi\} = 1^{[\Delta]} \in \mathrm{Ty}(\Delta)$ for each object $\Delta \in \mathcal{M}$ and morphism $\phi : \Delta \to \Gamma$;
- (o-Subst) $o_\Gamma\{\phi\} = o_\Delta \in \mathrm{Tm}(\Delta, 1^{[\Delta]})$.

Moreover, $\mathcal{M}$ *strictly supports One-type* if it additionally satisfies

- (o-Uniq) $\tau = o_\Gamma$ for all terms $\tau \in \mathrm{Tm}(\Gamma, 1^{[\Gamma]})$.[3]

[3]Note that o-Uniq implies Unit-Elim and Unit-Comp by $\mathcal{R}^1_A(a, t) := a$.

Convention. We often omit the superscript $[\Gamma]$ and/or subscript Γ when it does not bring confusion. This convention is applied to the other semantic type formers given below.

Definition 2.6 (semantic Zero-type [Hof97]). A CwF $\mathcal{M}$ *supports Zero-type* if

- (Zero-Form) There is a type $0^\Gamma \in \mathrm{Ty}(\Gamma)$ for each object $\Gamma \in \mathcal{M}$;
- (Zero-Elim) Given a type $A \in \mathrm{Ty}(\Gamma.0^{[\Gamma]})$ and a term $\zeta \in \mathrm{Tm}(\Gamma, 0^{[\Gamma]})$, there is a term $\mathcal{R}^0_A(\zeta) \in \mathrm{Tm}(\Gamma, A\{\overline{\zeta}\})$;
- (Zero-Subst) $0^{[\Gamma]}\{\phi\} = 0^{[\Delta]} \in \mathrm{Ty}(\Delta)$ for each object $\Delta \in \mathcal{M}$ and morphism $\phi : \Delta \to \Gamma$;
- ($\mathcal{R}^0$-Subst) $\mathcal{R}^0_{A\{\phi^+_0\}}(\zeta\{\phi\}) = \mathcal{R}^0_A(\zeta)\{\phi\}$.

Definition 2.7 (semantic Sum-type [Hof97]). A CwF $\mathcal{M}$ *supports Sum-type* if

- (Sum-form) There is a type $A + B \in \mathrm{Ty}(\Gamma)$ for each object $\Gamma \in \mathcal{M}$ and types $A, B \in \mathrm{Ty}(\Gamma)$;
- (Sum-intro) There are terms $\mathrm{inl}_{A,B} \in \mathrm{Tm}(\Gamma.A, A + B)$ and $\mathrm{inr}_{A,B} \in \mathrm{Tm}(\Gamma.B, A + B)$;
- (Sum-elim) Given a type $C \in \mathrm{Ty}(\Gamma.A + B)$ and morphisms $c_1 : \Gamma.A \to \Gamma.A + B.C$ and $c_2 : \Gamma.B \to \Gamma.A + B.C$ that satisfy the equations $\mathrm{p}_C \circ c_1 = \mathrm{inl}_{A,B}$ and $\mathrm{p}_C \circ c_2 = \mathrm{inr}_{A,B}$, there is a morphism $\mathcal{R}^+_C(c_1, c_2) : \Gamma.A + B \to \Gamma.A + B.C$;
- (Sum-comp) $\mathcal{R}^+_C(c_1, c_2) \circ \mathrm{inl}_{A,B} = c_1$ and $\mathcal{R}^+_C(c_1, c_2) \circ \mathrm{inl}_{A,B} = c_1$;
- (Sum-subst) $(A + B)\{\phi\} = A\{\phi\} + B\{\phi\}$ for each object $\Delta \in \mathcal{M}$ and morphism $\phi : \Delta \to \Gamma$;
- (Injection-subst) $\mathrm{inl}_{A,B} \circ \phi^+_A = \mathrm{inl}_{A\{\phi\},B\{\phi\}}$ and $\mathrm{inr}_{A,B} \circ \phi^+_A = \mathrm{inr}_{A\{\phi\},B\{\phi\}}$;
- ($\mathcal{R}^+$-subst) $\mathcal{R}^+_C(c_1, c_2) \circ \phi^+_{A+B} = \mathcal{R}^+_C(c_1 \circ \phi^+_1, c_2 \circ \phi^+_B)$.

Notation. We define a type $2 := 1 + 1 \in \mathrm{Ty}(\Gamma)$ for each object $\Gamma \in \mathcal{M}$, called the *boolean-type*, and morphisms $\underline{\mathrm{tt}}, \underline{\mathrm{ff}} : T \to 2$ by

$$\underline{\mathrm{tt}} := \mathrm{inl}_{1,1} \circ \langle \mathrm{id}, o\rangle \qquad \underline{\mathrm{ff}} := \mathrm{inr}_{1,1} \circ \langle \mathrm{id}, o\rangle.$$

Definition 2.8 (semantic Pi-types [Hof97]). A CwF $\mathcal{M}$ *supports Pi-types* if

- (Π-Form) Given an object $\Gamma \in \mathcal{M}$ and types $A \in \mathrm{Ty}(\Gamma)$ and $B \in \mathrm{Ty}(\Gamma.A)$, there is a type $\Pi(A, B) \in \mathrm{Ty}(\Gamma)$;
- (Π-Intro) Given a term $\beta \in \mathrm{Tm}(\Gamma.A, B)$, there is a term $\lambda_{A,B}(\beta) \in \mathrm{Tm}(\Gamma, \Pi(A, B))$;
- (Π-Elim) Given terms $\kappa \in \mathrm{Tm}(\Gamma, \Pi(A, B))$ and $\alpha \in \mathrm{Tm}(\Gamma, A)$, there is a term $\mathrm{App}_{A,B}(\kappa, \alpha) \in \mathrm{Tm}(\Gamma, B\{\overline{\alpha}\})$, where $\overline{\alpha} := \langle \mathrm{id}_\Gamma, \alpha\rangle_A : \Gamma \to \Gamma.A$;
- (Π-Comp) $\mathrm{App}_{A,B}(\lambda_{A,B}(\beta), \alpha) = \beta\{\overline{\alpha}\}$;
- (Π-Subst) $\Pi(A, B)\{\phi\} = \Pi(A\{\phi\}, B\{\phi^+_A\})$ for each object $\Delta \in \mathcal{M}$ and morphism $\phi : \Delta \to \Gamma$;
- (λ-Subst) $\lambda_{A,B}(\beta)\{\phi\} = \lambda_{A\{\phi\},B\{\phi^+_A\}}(\beta\{\phi^+_A\}) \in \mathrm{Tm}(\Delta, \Pi(A\{\phi\}, B\{\phi^+_A\}))$;
- (App-Subst) $\mathrm{App}_{A,B}(\kappa, \alpha)\{\phi\} = \mathrm{App}_{A\{\phi\},B\{\phi^+_A\}}(\kappa\{\phi\}, \alpha\{\phi\}) \in \mathrm{Tm}(\Delta, B\{\overline{\alpha}\}\{\phi\})$.

Moreover, $\mathcal{M}$ *strictly supports Pi-types* if it additionally satisfies

- (λ-Uniq) $\lambda_{A,B} \circ \mathrm{App}_{A\{\mathrm{p}_A\},B\{(\mathrm{p}_A)^+_{A\{\mathrm{p}_A\}}\}}(\kappa\{\mathrm{p}_A\}, \mathrm{v}_A) = \kappa$.

Notation. We write $A \Rightarrow B$ for $\Pi(A, B)$ if $B\{\langle \mathrm{id}_\Gamma, \alpha\rangle\} = B\{\langle \mathrm{id}_\Gamma, \alpha'\rangle\} \in \mathrm{Ty}(\Gamma)$ for all $\alpha, \alpha' \in \mathrm{Tm}(\Gamma, A)$.

Definition 2.9 (semantic Sigma-types [Hof97]). A CwF $\mathcal{M}$ *supports Sigma-types* if

- (Σ-Form) Given an object $\Gamma \in \mathcal{M}$ and types $A \in \mathrm{Ty}(\Gamma)$ and $B \in \mathrm{Ty}(\Gamma.A)$, there is a type $\Sigma(A, B) \in \mathrm{Ty}(\Gamma)$;

- (Σ-Intro) There is a morphism $\mathrm{Pair}_{A,B} : \Gamma.A.B \to \Gamma.\Sigma(A,B)$;
- (Σ-Elim) Given a type $P \in \mathrm{Ty}(\Gamma.\Sigma(A,B))$ and a term $\rho \in \mathrm{Tm}(\Gamma.A.B, P\{\mathrm{Pair}_{A,B}\})$, there is a term $\mathcal{R}^{\Sigma}_{A,B,P}(\rho) \in \mathrm{Tm}(\Gamma.\Sigma(A,B), P)$;
- (Σ-Comp) $\mathcal{R}^{\Sigma}_{A,B,P}(\rho)\{\mathrm{Pair}_{A,B}\} = \rho$;
- (Σ-Subst) $\Sigma(A,B)\{\phi\} = \Sigma(A\{\phi\}, B\{\phi^+_A\})$ for each object $\Delta \in \mathcal{M}$ and morphism $\phi : \Delta \to \Gamma$;
- (Pair-Subst) $\mathrm{p}_{\Sigma(A,B)} \circ \mathrm{Pair}_{A,B} = \mathrm{p}_A \circ \mathrm{p}_B$ and $\phi^+_{\Sigma(A,B)} \circ \mathrm{Pair}_{A\{\phi\},B\{\phi^+_A\}} = \mathrm{Pair}_{A,B} \circ \phi^{++}_{A,B}$;
- ($\mathcal{R}^{\Sigma}$-Subst) $\mathcal{R}^{\Sigma}_{A,B,P}(p)\{\phi^+_{\Sigma(A,B)}\} = \mathcal{R}^{\Sigma}_{A\{f\},B\{\phi^+_A\},P\{\phi^+_{\Sigma(A,B)}\}}(p\{\phi^{++}_{A,B}\})$.

Moreover, $\mathcal{M}$ *strictly supports Sigma-types* if it additionally satisfies

- ($\mathcal{R}^{\Sigma}$-Uniq) $\check{\rho} = \mathcal{R}^{\Sigma}_{A,B,P}(\rho)$ if $\check{\rho} \in \mathrm{Tm}(\Gamma.\Sigma(A,B), P)$ and $\check{\rho}\{\mathrm{Pair}_{A,B}\} = \rho$.

Notation. We write $A \times B$ for $\Sigma(A,B)$ if $B\{\langle \mathrm{id}_\Gamma, \alpha\rangle\} = B\{\langle \mathrm{id}_\Gamma, \alpha'\rangle\} \in \mathrm{Ty}(\Gamma)$ for all $\alpha, \alpha' \in \mathrm{Tm}(\Gamma, A)$.

Definition 2.10 (semantic N-type [Hof97])**.** A CwF $\mathcal{M}$ *supports N-type* if

- (N-Form) There is a type $N^{[\Gamma]} \in \mathrm{Ty}(\Gamma)$ for each object $\Gamma \in \mathcal{M}$;
- (N-Intro) There are a term $\underline{0}_\Gamma \in \mathrm{Tm}(\Gamma, N)$ and a morphism $\mathrm{succ}_\Gamma : \Gamma.N \to \Gamma.N$ that satisfy, for all morphisms $\phi : \Delta \to \Gamma$ and $\psi : \Delta.N \to \Gamma$, the equations

$$\underline{0}_\Gamma\{\phi\} = \underline{0}_\Delta \in \mathrm{Tm}(\Delta, N) \qquad \mathrm{p}_N \circ \mathrm{succ}_\Gamma = \mathrm{p}_N : \Gamma.N \to \Gamma$$

$$\mathrm{succ}_\Gamma \circ \langle \psi, \mathrm{v}_N\rangle_N = \langle \psi, \mathrm{v}_N\{\mathrm{succ}_\Delta\}\rangle_N : \Delta.N \to \Gamma.N,$$

  where the last equation makes sense since $\mathrm{succ}_\Gamma \circ \langle \psi, \mathrm{v}_N\rangle_N, \langle \psi, \mathrm{v}_N\{\mathrm{succ}_\Delta\}\rangle_N : \Delta.N \to \Gamma.N$ by N-Subst given below. We henceforth skip the same remark.

  Notation. Let $\mathrm{zero}_\Gamma := \langle \mathrm{id}_\Gamma, \underline{0}_\Gamma\rangle_N : \Gamma \to \Gamma.N$ for each object $\Gamma \in \mathcal{C}$, which satisfies the equation $\mathrm{zero}_\Gamma \circ \phi = \langle \phi, \underline{0}_\Delta\rangle_N = \langle \phi, \mathrm{v}_N\{\mathrm{zero}_\Delta\}\rangle_N : \Delta \to \Gamma.N$ for each morphism $\phi : \Delta \to \Gamma$. We then define a term $\underline{n}_\Gamma \in \mathrm{Tm}(\Gamma, N)$ for each $n \in \mathbb{N}$: $\underline{0}_\Gamma$ is already given, and $\underline{n+1}_\Gamma := \mathrm{v}_N\{\mathrm{succ}_\Gamma \circ \langle \mathrm{id}_\Gamma, \underline{n}_\Gamma\rangle_N\}$.
- (N-Elim) Given a type $P \in \mathrm{Ty}(\Gamma.N)$ and terms $c_{\mathrm{z}} \in \mathrm{Tm}(\Gamma, P\{\mathrm{zero}\})$ and $c_{\mathrm{s}} \in \mathrm{Tm}(\Gamma.N.P, P\{\mathrm{succ} \circ \mathrm{p}_P\})$, there is a term $\mathcal{R}^N_P(c_{\mathrm{z}}, c_{\mathrm{s}}) \in \mathrm{Tm}(\Gamma.N, P)$;
- (N-Comp) We have the equations

$$\mathcal{R}^N_P(c_{\mathrm{z}}, c_{\mathrm{s}})\{\mathrm{zero}\} = c_{\mathrm{z}} \in \mathrm{Tm}(\Gamma, P\{\mathrm{zero}\});$$
$$\mathcal{R}^N_P(c_{\mathrm{z}}, c_{\mathrm{s}})\{\mathrm{succ}\} = c_{\mathrm{s}}\{\langle \mathrm{pId}_{\Gamma.N}, \mathcal{R}^N_P(c_{\mathrm{z}}, c_{\mathrm{s}})\rangle_P\} \in \mathrm{Tm}(\Gamma.N, P\{\mathrm{succ}\});$$

- (N-Subst) $N^{[\Gamma]}\{\phi\} = N^{[\Delta]} \in \mathrm{Ty}(\Delta)$;
- ($\mathcal{R}^N$-Subst) $\mathcal{R}^N_P(c_{\mathrm{z}}, c_{\mathrm{s}})\{\phi^+_N\} = \mathcal{R}^N_{P\{\phi^+_N\}}(c_{\mathrm{z}}\{\phi\}, c_{\mathrm{s}}\{\phi^{++}_{N,P}\}) \in \mathrm{Tm}(\Delta.N, P\{\phi^+_N\})$.

Definition 2.11 (semantic extensional Id-types [Hof97])**.** A CwF $\mathcal{M}$ *supports Id-types* if

- (Id-Form) Given an object $\Gamma \in \mathcal{M}$ and a type $A \in \mathrm{Ty}(\Gamma)$, there is a type $\mathrm{Id}_A \in \mathrm{Ty}(\Gamma.A.A^+)$, where $A^+ := A\{\mathrm{p}_A\} \in \mathrm{Ty}(\Gamma.A)$;
- (Id-Intro) There is a morphism $\mathrm{Refl}_A : \Gamma.A \to \Gamma.A.A^+.\mathrm{Id}_A$ that satisfies the equation $\mathrm{p}_{\mathrm{Id}_A} \circ \mathrm{Refl}_A = \overline{\mathrm{v}_A} : \Gamma.A \to \Gamma.A.A^+$, where $\overline{\mathrm{v}_A} := \langle \mathrm{Id}_{\Gamma.A}, \mathrm{v}_A\rangle$;
- (Id-Elim) Given a type $B \in \mathrm{Ty}(\Gamma.A.A^+.\mathrm{Id}_A)$ and a term $\beta \in \mathrm{Tm}(\Gamma.A, B\{\mathrm{Refl}_A\})$, there is a term $\mathcal{R}^{\mathrm{Id}}_{A,B}(\beta) \in \mathrm{Tm}(\Gamma.A.A^+.\mathrm{Id}_A, B)$;
- (Id-Comp) $\mathcal{R}^{\mathrm{Id}}_{A,B}(\beta)\{\mathrm{Refl}_A\} = \beta$;

- (Id-Subst) $\mathrm{Id}_A\{\phi^{++}_{A,A^+}\} = \mathrm{Id}_{A\{\phi\}} \in \mathrm{Ty}(\Delta.A\{\phi\}.A\{\phi\}^+)$ for each object $\Delta \in \mathcal{M}$ and morphism $\phi : \Delta \to \Gamma$;
- (Refl-Subst) $\mathrm{Refl}_A \circ \phi^+_A = \phi^{+++}_{A,A^+,\mathrm{Id}_A} \circ \mathrm{Refl}_{A\{\phi\}} : \Delta.A\{\phi\} \to \Gamma.A.A^+.\mathrm{Id}_A$;
- ($\mathcal{R}^{\mathrm{Id}}$-Subst) $\mathcal{R}^{\mathrm{pId}}_{A,B}(\beta)\{\phi^{+++}_{A,A^+,\mathrm{Id}_A}\} = \mathcal{R}^{\mathrm{Id}}_{A\{\phi\},B\{\phi^{+++}_{A,A^+,\mathrm{Id}_A}\}}(\beta\{\phi^+_A\})$.

Moreover, Id-types are *propositional* if the set $\mathrm{Tm}(\Gamma, \mathrm{pId}_A\{\langle\langle \mathrm{id}_\Gamma, \alpha\rangle, \alpha'\rangle\})$ has at most one inhabitant for each object $\Gamma \in \mathcal{M}$, type $A \in \mathrm{Ty}(\Gamma)$ and terms $\alpha, \alpha' \in \mathrm{Tm}(\Gamma, A)$. In this case, we write pId for Id.

Definition 2.12 (semantic universes). A CwF $\mathcal{M}$ *supports universes* if

- (U-Form) Given an object $\Gamma \in \mathcal{M}$, there is a type $\mathcal{U}^{[\Gamma]}_k \in \mathrm{Ty}(\Gamma)$ for each $k \in \mathbb{N}$;
- (U-Intro) Given a type $A \in \mathrm{Ty}(\Gamma)$, there is a term $\mathrm{En}(A) \in \mathrm{Tm}(\Gamma, \mathcal{U}_k)$ for some $k \in \mathbb{N}$, and in particular $\mathrm{En}(\mathcal{U}_l) \in \mathrm{Tm}(\Gamma, \mathcal{U}_{l+1})$ for all $l \in \mathbb{N}$;
- (U-Elim) Each term $c \in \mathrm{Tm}(\Gamma, \mathcal{U})$ induces a type $\mathrm{El}(c) \in \mathrm{Ty}(\Gamma)$;
- (U-Comp) $\mathrm{El}(\mathrm{En}(A)) = A$;
- (U-Cumul) If $c \in \mathrm{Tm}(\Gamma, \mathcal{U}_k)$, then $c \in \mathrm{Tm}(\Gamma, \mathcal{U}_{k+1})$;
- (U-Subst) $\mathcal{U}^{[\Gamma]}_k\{\phi\} = \mathcal{U}^{[\Delta]}_k \in \mathrm{Ty}(\Delta)$ for each natural number $k \in \mathbb{N}$ and morphism $\phi : \Delta \to \Gamma$;
- (En-Subst) $\mathrm{En}(A)\{\phi\} = \mathrm{En}(A\{\phi\}) \in \mathrm{Tm}(\Delta, \mathcal{U})$.

If the CwF $\mathcal{M}$ is standard, then universes $\mathcal{U}_k$ are *contextual* if they are equipped with terms $\ell_k \in \mathrm{Tm}(T.\mathcal{U}_k, N)$ for all $k \in \mathbb{N}$ that internalise the length function lth in the sense that the equation

$$\ell_k\{\langle !, \mu\rangle\} = \underline{\mathrm{lth} \circ \iota^{-1} \circ \mathrm{El}(\mu)} \in \mathrm{Tm}(T, N) \tag{2}$$

holds for each term $\mu \in \mathrm{Tm}(T, \mathcal{U}_k)$.

3 Internal ∞-groupoids

As announced in §1.3, we interpret HoTT via CwFs recalled in the previous section. Concretely we first define a CwF $\infty\mathcal{M}$Gpd of ∞-groupoids internal to a standard, locally small CwF $\mathcal{M}$ equipped with universes, extensional Id-types, Pi-types and N-type in §3–4, and then focus on the CwF $\infty\mathcal{G}$Gpd of ∞-groupoids internal to the CwF $\mathcal{G}$ of games equipped with these types in §5. The two conditions and the four types on $\mathcal{M}$ play crucial roles for the construction of $\infty\mathcal{M}$Gpd. The goal of this section is to describe the underlying category of $\infty\mathcal{M}$Gpd, leaving its other parts to the next section. We fix a standard, locally small CwF $\mathcal{M}$ equipped with universes, extensional Id-types, Pi-types and N-type.

This section proceeds as follows. In §3.1, we internalise *(strict) ∞-categories*, *(strict) ∞-functors* and *(strict) ∞-natural transformations* to $\mathcal{M}$, and define an ∞-category $\infty\mathcal{M}$Cat of these structures. Next, in §3.2, we define *internal ∞-groupoids* to be the ∞-categories internal to $\mathcal{M}$ together with *weak* inverses and their *witnesses*, and focus on the ∞-functors internal to $\mathcal{M}$ that respect the ∞-groupoid structure. We finally show that these data form an ∞-groupoid $\infty\mathcal{M}\mathrm{Gpd} \hookrightarrow \infty\mathcal{M}\mathrm{Cat}$ and based on this result define *$\mathcal{M}$-internal dependent ∞-groupoids*, our interpretation of dependent types, in §3.3.

Notation. We employ the following notations.

1. We also write $\mathrm{d} : \mathrm{Tm}(T, \mathrm{d}(\Delta) \Rightarrow \mathrm{d}(\Gamma)^+) \xrightarrow{\sim} \mathcal{M}(\Delta, \Gamma)$ $(\Delta, \Gamma \in \mathcal{M})$ for the bijection given by $\mathrm{d}(f) := \iota^{-1} \circ \langle !, \lambda^{-1}(f)\rangle \circ \iota : \Delta \to \Gamma$ $(f \in \mathrm{Tm}(1, \mathrm{d}(\Delta) \Rightarrow \mathrm{d}(\Gamma)^+))$.
2. We define a type $N^{[\Gamma]}_b(n) \in \mathrm{Ty}(\Gamma)$ for each $\Gamma \in \mathcal{M}$ and $n \in \mathbb{N}$ by

$$N^{[\Gamma]}_b(n) := \underbrace{1^{[\Gamma]} + 1^{[\Gamma]} + \cdots + 1^{[\Gamma]}}_{n} \tag{3}$$

Clearly, $N_b^{[\Gamma]}(n)$ has the canonical terms that correspond to the natural numbers p such that $p \leqslant n$. We abuse notation and write $\underline{p} \in \mathrm{Tm}(\Gamma, N_b^{[\Gamma]})$ for these terms. Finally, we define a type $N_b^{[\Gamma]} \in \mathrm{Ty}(\Gamma.N^{[\Gamma]})$ by applying the elimination rule of N-type $N^{[\Gamma]}$ with respect to the first universe $\mathcal{U}_0$ in the evident way that satisfies

$$N_b^{[T]}\{\langle \mathrm{id}_T, \underline{n}\rangle\} = \mathrm{d}(N_b^{[T]}(n)) \tag{4}$$

for all $n \in \mathbb{N}$. We often abbreviate the superscript $(\_)^{[\Gamma]}$ on $N_b^{[\Gamma]}(n)$ and $N_b^{[\Gamma]}$ when it does not bring confusion.

3. We write Γ^n for the left-associative product $\underbrace{\Gamma \times \Gamma \times \cdots \times \Gamma}_{n}$, and given a left-associative product $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ and an index $i \in \{1, 2, \ldots, n\}$ we write $\pi_i^{[n]}$ or $\pi_i^{[n]}(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n)$ for the iterated projection $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n \to \Gamma_i$ defined by $\pi_i^{[n]} := \pi_2 \circ \pi_1^{n-i}$ if $i > 1$ and $\pi_1^{[n]} := \pi_1^n$. Similarly, given a type $A \in \mathrm{Ty}(\Gamma)$, we write $\Gamma.A^n$ for the comprehension $\Gamma.A.A\{\mathrm{p}\} \ldots A\{\mathrm{p}^{n-1}\}$, and given a comprehension $\Gamma.A_1.A_2 \ldots A_n$ we define a term $\pi_i^{[n+1]} \in \mathrm{Tm}(\Gamma.A_1.A_2 \ldots A_n, A_i\{\mathrm{p}^{n-i}\})$ by $\pi_i^{[n]} := \mathrm{v}\{\mathrm{p}^{n-i}\}$ if $1 < i \leqslant n+1$, and a morphism $\pi_1^{[n+1]} : \Gamma.A_1.A_2 \ldots A_n \to \Gamma$ by $\pi_1^{[n+1]} := \mathrm{p}^n$.

3.1 Internal ∞-categories, ∞-functors and ∞-transformations

We first introduce internal ∞-categories, ∞-functors and ∞-natural transformations. The idea of *internal categories* [ML13, §12.1] is to regard small categories as formalised within the ambient category Set of sets and generalise them to those formalised within an arbitrary category $\mathcal{P}$ with pullbacks, called *categories internal to* $\mathcal{P}$ or $\mathcal{P}$*-internal categories*, where pullbacks are for internally reformulating the composition of morphisms. It is then just a routine to accordingly internalise functors and natural transformations to $\mathcal{P}$.

Remark. The game-semantic CwF $\mathcal{G}$ (§5) does not have all pullbacks, but it has the pullbacks necessary to internalise the composition of morphisms (Definition 3.1). This is why we do not assume that the arbitrary CwF $\mathcal{M}$ has all pullbacks. In other words, the traditional assumption for internal categories that the ambient category has all pullbacks is too strong to subsume the game-semantic instance.

In the same vein, it is also straightforward to internalise *(strict)* ∞*-categories*, *(strict)* ∞*-functors* and *(strict)* ∞*-natural transformations* (à la *globular sets* [Ehr64, Lei04]):

Definition 3.1 (internal ∞-categories)**.** An $\mathcal{M}$***-internal*** ∞***-category*** $\mathcal{C}$ consists of

- A type $\mathcal{C} \in \mathrm{Ty}(T.N)$ and terms $\mathrm{s}, \mathrm{t} \in \mathrm{Tm}(T.N, \mathcal{C}\{\mathrm{succ}\} \Rightarrow \mathcal{C}^+)$ in $\mathcal{M}$ that satisfy the so-called ***globular equations*** between the morphisms

$$\mathrm{s}_n \circ \mathrm{s}_{n+1} = \mathrm{s}_n \circ \mathrm{t}_{n+1} : T.\mathcal{C}_{n+2} \to T.\mathcal{C}_n \qquad \mathrm{t}_n \circ \mathrm{s}_{n+1} = \mathrm{t}_n \circ \mathrm{t}_{n+1} : T.\mathcal{C}_{n+2} \to T.\mathcal{C}_n$$

  in $\mathcal{M}$ for all $n \in \mathbb{N}$, where $\mathcal{C}_n := \mathcal{C}\{\langle !, \underline{n}\rangle\} \in \mathrm{Ty}(T)$, $\mathrm{s}_n := \mathrm{d}(\mathrm{s}\{\langle !, \underline{n}\rangle\}) : T.\mathcal{C}_{n+1} \to T.\mathcal{C}_n$ and $\mathrm{t}_n := \mathrm{d}(\mathrm{t}\{\langle !, \underline{n}\rangle\}) : T.\mathcal{C}_{n+1} \to T.\mathcal{C}_n$, and we often omit the subscript $(\_)_n$ on s and t;

- A term $* \in \mathrm{Tm}(T.N.N_b, \mathrm{Comp})$ in $\mathcal{M}$, where $\mathrm{Comp} \in \mathrm{Ty}(T.N.N_b)$ is the (unique by the democracy of $\mathcal{M}$ and the equation below) type obtained by the elimination rules of sum- and N-types with respect to a universe $\mathcal{U}$ large enough to encode $\mathcal{C}_n$ such that

$$T.\mathrm{Comp}\{\langle\langle !, \underline{n}\rangle, \underline{p}\rangle\} = T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n \Rightarrow T.\mathcal{C}_n$$

  for all $n, p \in \mathbb{N}$ such that $n > p$, and $T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n$ is the chosen pullback[4]

$$\begin{array}{ccc} T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n & \xrightarrow{\varpi_2} & T.\mathcal{C}_n \\ {\scriptstyle \varpi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{t}^{n-p}} \\ T.\mathcal{C}_n & \xrightarrow[\mathrm{s}^{n-p}]{} & T.\mathcal{C}_p \end{array}$$

[4]We leave it to the reader to verify that the following diagram in fact forms a pullback.

given by

$$T.\mathcal{C}_n \times_{\mathcal{C}_p} T.\mathcal{C}_n := T.\mathcal{C}_n.\mathcal{C}_n^+.\mathrm{pId}_{\mathcal{C}_p}\{\langle \mathrm{s}^{n-p} \circ \mathrm{p}, \mathrm{v}\{\mathrm{t}^{n-p} \circ \langle !, \mathrm{v}\rangle\}\rangle\}$$

$$\varpi_1 := \mathrm{p} \circ \mathrm{p} \qquad\qquad \varpi_2 := \langle !, \mathrm{v}\{\mathrm{p}\}\rangle,$$

such that the morphism

$$*_p^{[n]} := \mathrm{d}(*\{\langle\langle !, \underline{n}\rangle, \underline{p}\rangle\}) : T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n \to T.\mathcal{C}_n$$

in $\mathcal{M}$ satisfies the equations[5]

$$\begin{aligned}
\mathrm{s}_p \circ *_p^{[p+1]} &= \mathrm{s}_p \circ \varpi_2 : T.\mathcal{C}_{p+1} \times_{T.\mathcal{C}_p} T.\mathcal{C}_{p+1} \to T.\mathcal{C}_p \\
\mathrm{t}_p \circ *_p^{[p+1]} &= \mathrm{t}_p \circ \varpi_1 : T.\mathcal{C}_{p+1} \times_{T.\mathcal{C}_p} T.\mathcal{C}_{p+1} \to T.\mathcal{C}_p \\
\mathrm{s}_n \circ *_p^{[n+1]} &= *_p^{[n]} \circ (\mathrm{s}_n \times_{T.\mathcal{C}_p} \mathrm{s}_n) : T.\mathcal{C}_{n+1} \times_{T.\mathcal{C}_p} T.\mathcal{C}_{n+1} \to T.\mathcal{C}_n \\
\mathrm{t}_n \circ *_p^{[n+1]} &= *_p^{[n]} \circ (\mathrm{t}_n \times_{T.\mathcal{C}_p} \mathrm{t}_n) : T.\mathcal{C}_{n+1} \times_{T.\mathcal{C}_p} T.\mathcal{C}_{n+1} \to T.\mathcal{C}_n,
\end{aligned}$$

where we often omit the subscript $(\_)_p$ and/or the superscript $(\_)^{[n]}$ on $*$;

- A term $\mathrm{i} \in \mathrm{Tm}(T.N, \mathcal{C} \Rightarrow \mathcal{C}\{\mathrm{succ}\}^+)$ in $\mathcal{M}$ such that the morphism $\mathrm{i}_n := \mathrm{d}(\mathrm{i}\{\langle !, \underline{n}\rangle\}) : T.\mathcal{C}_n \to T.\mathcal{C}_{n+1}$ in $\mathcal{M}$ is a section of s_n and t_n for all $n \in \mathbb{N}$, where we often omit the subscript $(\_)_n$ on i,

and these structures satisfy

1. (Associativity of compositions) Given $n, p \in \mathbb{N}$ such that $n > p$, the equation

$$*_p^{[n]} \circ (*_p^{[n]} \times_{T.\mathcal{C}_p} \mathrm{id}_{T.\mathcal{C}_n}) = *_p^{[n]} \circ (\mathrm{id}_{T.\mathcal{C}_n} \times_{T.\mathcal{C}_p} *_p^{[n]}) \circ \alpha : (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) \times_{T.\mathcal{C}_p} T.\mathcal{C}_n \to T.\mathcal{C}_n,$$

where the isomorphism $\alpha : (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) \times_{T.\mathcal{C}_p} T.\mathcal{C}_n \xrightarrow{\sim} T.\mathcal{C}_n \times_{T.\mathcal{C}_p} (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n)$ is given by

$$\alpha := \langle \varpi_1 \circ \varpi_1, \langle \varpi_2 \circ \varpi_1, \varpi_2\rangle_{T.\mathcal{C}_p}\rangle_{T.\mathcal{C}_p},$$

and $(T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) \times_{T.\mathcal{C}_p} T.\mathcal{C}_n$ and $T.\mathcal{C}_n \times_{T.\mathcal{C}_p} (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n)$ are the pullbacks

$$\begin{array}{ccc}
(T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) \times_{T.\mathcal{C}_p} T.\mathcal{C}_n & \xrightarrow{\varpi_2} & T.\mathcal{C}_n \\
{\scriptstyle \varpi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{t}^{n-p}} \\
T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n & \xrightarrow[\mathrm{s}^{n-p} \circ \varpi_2]{} & T.\mathcal{C}_p
\end{array}
\qquad
\begin{array}{ccc}
T.\mathcal{C}_n \times_{T.\mathcal{C}_p} (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) & \xrightarrow{\varpi_2} & T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n \\
{\scriptstyle \varpi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{t}^{n-p} \circ \varpi_1} \\
T.\mathcal{C}_n & \xrightarrow[\mathrm{s}^{n-p}]{} & T.\mathcal{C}_p
\end{array}$$

2. (Unit laws) Given $n, p \in \mathbb{N}$ such that $n > p$, the equation

$$*_p^{[n]} \circ \langle \mathrm{id}_{T.\mathcal{C}_n}, \mathrm{i}^{n-p} \circ \mathrm{s}^{n-p}\rangle_{T.\mathcal{C}_p} = \mathrm{id}_{T.\mathcal{C}_n} = *_p^{[n]} \circ \langle \mathrm{i}^{n-p} \circ \mathrm{t}^{n-p}, \mathrm{id}_{T.\mathcal{C}_n}\rangle_{T.\mathcal{C}_p} : T.\mathcal{C}_n \to T.\mathcal{C}_n;$$

3. (Interchange laws) Given $n, p, q \in \mathbb{N}$ such that $p < q < n$, the equations

$$\begin{aligned}
&*_q^{[n]} \circ (*_p^{[n]} \times_{T.\mathcal{C}_q} *_p^{[n]}) = *_p^{[n]} \circ (*_q^{[n]} \times_{T.\mathcal{C}_p} *_q^{[n]}) \circ \chi : (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} \mathcal{C}_n) \times_{T.\mathcal{C}_q} (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) \to T.\mathcal{C}_n \\
&*_p^{[n+1]} \circ (\mathrm{i}_n \times_{T.\mathcal{C}_p} \mathrm{i}_n) = \mathrm{i}_n \circ *_p^{[n]} : T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n \to T.\mathcal{C}_{n+1},
\end{aligned}$$

where the isomorphism $\chi : (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) \times_{T.\mathcal{C}_q} (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) \xrightarrow{\sim} (T.\mathcal{C}_n \times_{T.\mathcal{C}_q} T.\mathcal{C}_n) \times_{T.\mathcal{C}_p} (T.\mathcal{C}_n \times_{T.\mathcal{C}_q} T.\mathcal{C}_n)$ is given by

$$\chi := \langle\langle \varpi_1 \circ \varpi_1, \varpi_1 \circ \varpi_2\rangle_{T.\mathcal{C}_q}, \langle \varpi_2 \circ \varpi_1, \varpi_2 \circ \varpi_2\rangle_{T.\mathcal{C}_q}\rangle_{T.\mathcal{C}_p},$$

[5] The morphisms $\mathrm{s}_n \times_{T.\mathcal{C}_p} \mathrm{s}_n, \mathrm{t}_n \times_{T.\mathcal{C}_p} \mathrm{t}_n : T.\mathcal{C}_{n+1} \times_{T.\mathcal{C}_p} T.\mathcal{C}_{n+1} \to T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n$ are well-defined as the equations $\mathrm{t}^{n-p} \circ \mathrm{s}_n = \mathrm{t}^{n+1-p} : T.\mathcal{C}_{n+1} \to T.\mathcal{C}_p$ and $\mathrm{s}^{n-p} \circ \mathrm{t}_n = \mathrm{s}^{n+1-p} : T.\mathcal{C}_{n+1} \to T.\mathcal{C}_p$ hold thanks to the globular equations. We henceforth skip making a similar remark.

and $(T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) \times_{T.\mathcal{C}_q} (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n)$ and $(T.\mathcal{C}_n \times_{T.\mathcal{C}_q} T.\mathcal{C}_n) \times_{T.\mathcal{C}_p} (T.\mathcal{C}_n \times_{T.\mathcal{C}_q} T.\mathcal{C}_n)$ are the pullbacks

$$\begin{array}{ccc} (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) \times_{T.\mathcal{C}_q} (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) & \xrightarrow{\varpi_2} & (T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n) \\ {\scriptstyle \varpi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{t}^{n-q} \circ *_p} \\ T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n & \xrightarrow[\mathrm{s}^{n-q} \circ *_p]{} & T.\mathcal{C}_q \end{array}$$

$$\begin{array}{ccc} (T.\mathcal{C}_n \times_{T.\mathcal{C}_q} T.\mathcal{C}_n) \times_{T.\mathcal{C}_p} (T.\mathcal{C}_n \times_{T.\mathcal{C}_q} T.\mathcal{C}_n) & \xrightarrow{\varpi_2} & (T.\mathcal{C}_n \times_{T.\mathcal{C}_q} T.\mathcal{C}_n) \\ {\scriptstyle \varpi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{t}^{n-p} \circ *_q} \\ T.\mathcal{C}_n \times_{T.\mathcal{C}_q} T.\mathcal{C}_n & \xrightarrow[\mathrm{s}^{n-p} \circ *_q]{} & T.\mathcal{C}_p \end{array}$$

Convention. We call generalised elements $\sigma : \Gamma \to T.\mathcal{C}_n$ ***n-cells*** in an $\mathcal{M}$-internal ∞-category $\mathcal{C}$. Further, we call s ***sources***, t ***targets***, $*$ ***compositions***, and i ***identities*** in $\mathcal{C}$.

Example 3.2. As a canonical example, consider the case of $\mathcal{M} = \mathrm{Set}$. Set-internal ∞-categories are precisely ∞-categories in the ordinary sense [War11, §2.1], where our notations and terminologies suggest which structures correspond to each other. In this sense, $\mathcal{M}$-internal ∞-categories generalise ∞-categories.

Notation. Let $\alpha, \beta : \Gamma \to T.\mathcal{C}_n$ and $\sigma, \tau : \Gamma \to T.\mathcal{C}_{n+1}$ be cells in an $\mathcal{M}$-internal ∞-category $\mathcal{C}$, and $\mathrm{s} \circ \sigma = \alpha = \mathrm{s} \circ \tau$ and $\mathrm{t} \circ \sigma = \beta = \mathrm{t} \circ \tau$. Following Warren [War11, §2.2], we write

$$\alpha \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \beta \quad (\phi)$$

for a cell ϕ in $\mathcal{C}$ ***bounded*** by σ and τ, i.e., a morphism $\phi : \Gamma \to T.\mathcal{C}_m$ in $\mathcal{M}$ for some $m > n$ that satisfies $\mathrm{s}^{m-(n+1)} \circ \phi = \sigma$ and $\mathrm{t}^{m-(n+1)} \circ \phi = \tau$ (n.b., $\sigma = \phi = \tau$ if $m = n + 1$).

Given an ordinary ∞-category $\mathcal{S}$ and n-cells $\mu, \nu \in \mathcal{S}_n$ ($n \in \mathbb{N}$), one may take the evident substructural ∞-category $\mathcal{S}_{n+1}(\mu, \nu) \hookrightarrow \mathcal{S}$ whose m-cells ($m \in \mathbb{N}$) are $(m+n+1)$-cells ϕ in $\mathcal{S}$ that satisfy $\mathrm{s}^{m+1}(\phi) = \mu$ and $\mathrm{t}^{m+1}(\phi) = \nu$ [War11, §2.2]. We generalise this construction to:

Notation. Given global n-cells $\sigma, \tau : T \to T.\mathcal{C}_n$ in an $\mathcal{M}$-internal ∞-category $\mathcal{C}$, we define an type $\mathcal{C}_{n+1}[\sigma, \tau]_m \in \mathrm{Ty}(T)$ for each $m \in \mathbb{N}$ by

$$\mathcal{C}_{n+1}[\sigma, \tau]_m := \mathrm{d}(T.\mathcal{C}_{n+1+m}.\mathrm{pId}_{\mathcal{C}_n \times \mathcal{C}_n}\{\langle\langle !, \mathrm{v}\{\langle \mathrm{s}^{m+1}, \mathrm{t}^{m+1}\rangle\}\rangle, \mathrm{v}\{\langle \sigma, \tau\rangle \circ !\}\rangle\}).$$

By the elimination rule of N-type with respect to a sufficiently large universe, these types $\mathcal{C}_{n+1}[\sigma, \tau]_m$ for all $m \in \mathbb{N}$ give rise to a type $\mathcal{C}_{n+1}[\sigma, \tau] \in \mathrm{Ty}(T.N)$. Moreover, it is straightforward to lift this type to the substructural $\mathcal{M}$-internal ∞-category $\mathcal{C}_{n+1}[\sigma, \tau]$ of $\mathcal{C}$ whose other structures inherit from the corresponding ones of $\mathcal{C}$ in the evident way.

Clearly, the two constructions of $\mathcal{S}_{n+1}(\sigma, \tau)$ and $\mathcal{C}_{n+1}[\sigma, \tau]$ coincide when $\mathcal{M} = \mathrm{Set}$.

Remark. At first glance it seems that the object $T.\mathcal{C}_{n+1}[\sigma, \tau]_m$ forms an equaliser

$$T.\mathcal{C}_{n+1}[\sigma, \tau]_m \hookrightarrow T.\mathcal{C}_{n+m+1} \underset{!;\langle \sigma, \tau\rangle}{\overset{\langle \mathrm{s}^{m+1}, \mathrm{t}^{m+1}\rangle}{\rightrightarrows}} T.\mathcal{C}_n \times T.\mathcal{C}_n$$

However, it is *not* the case for the game-semantic CwF $\mathcal{G}$; see §5 for the details. Thus, for the present work, we cannot replace the CwF $\mathcal{M}$ with a finitely complete category.

Definition 3.3 (internal ∞-functors)**.** An ***$\mathcal{M}$-internal ∞-functor*** between $\mathcal{M}$-internal ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is a term $F \in \mathrm{Tm}(T.N, \mathcal{C} \Rightarrow \mathcal{D}^+)$ in $\mathcal{M}$, written $F : \mathcal{C} \to \mathcal{D}$, such that the family $\mathrm{fun}(F) := (F_n := \mathrm{d}(F\{\langle !, \underline{n}\rangle\}))_{n \in \mathbb{N}}$ of morphisms $F_n : T.\mathcal{C}_n \to T.\mathcal{D}_n$ in $\mathcal{M}$ satisfies

1. (Preservation of sources and targets) For all $n \in \mathbb{N}$, the equations

$$\mathrm{s}_n \circ F_{n+1} = F_n \circ \mathrm{s}_n : T.\mathcal{C}_{n+1} \to T.\mathcal{D}_n \qquad \mathrm{t}_n \circ F_{n+1} = F_n \circ \mathrm{t}_n : T.\mathcal{C}_{n+1} \to T.\mathcal{D}_n;$$

2. (Preservation of compositions) For all $n, p \in \mathbb{N}$ such that $n > p$, the equation

$$F_n \circ *_p^{[n]} = *_p^{[n]} \circ (F_n \times_{T.\mathcal{D}_p} F_n) : T.\mathcal{C}_n \times_{T.\mathcal{C}_p} T.\mathcal{C}_n \to T.\mathcal{D}_n;$$

3. (Preservation of identities) For all $n \in \mathbb{N}$, the equation

$$F_{n+1} \circ i_n = i_n \circ F_n : T.\mathcal{C}_n \to T.\mathcal{D}_{n+1}.$$

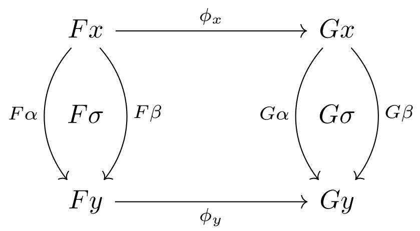
Example 3.4. Set-internal ∞-functors F are precisely ∞-functors [War11, §2.1], and they are equivalent to the induced families $\mathrm{fun}(F)$ of morphisms. In this sense, $\mathcal{M}$-internal ∞-functors generalise ∞-functors.

Definition 3.5 (internal ∞-natural transformations)**.** An ***$\mathcal{M}$-internal ∞-natural (1-) transformation*** between $\mathcal{M}$-internal ∞-functors $F, G : \mathcal{C} \to \mathcal{D}$ (or ***bounded*** by F and G) is a morphism $\phi : T.\mathcal{C}_0 \to T.\mathcal{D}_1$ in $\mathcal{M}$, written $\phi : F \to G$, that satisfies

1. (Source and target) $\mathrm{s} \circ \phi = F_0 : T.\mathcal{C}_0 \to T.\mathcal{D}_0$ and $\mathrm{t} \circ \phi = G_0 : T.\mathcal{C}_0 \to T.\mathcal{D}_0$;
2. (Naturality) $*_0 \circ \langle G_{m+1}, \mathrm{i}^m \circ \phi \circ \mathrm{s}^{m+1} \rangle_{T.\mathcal{D}_0} = *_0 \circ \langle \mathrm{i}^m \circ \phi \circ \mathrm{t}^{m+1}, F_{m+1} \rangle_{T.\mathcal{D}_0} : T.\mathcal{C}_{m+1} \to T.\mathcal{D}_{m+1}$ for all $m \in \mathbb{N}$.

Example 3.6. Set-internal ∞-natural transformations ϕ between Set-internal ∞-functors $F, G : \mathcal{C} \to \mathcal{D}$ are precisely ∞-natural transformations between F and G [War11, §2.1]. In this sense, $\mathcal{M}$-internal ∞-natural transformations generalise ∞-natural transformations.

When $\mathcal{M} = \mathrm{Set}$, the naturality of ϕ corresponds to the familiar commutativity of the diagram



in Set for all $m \in \mathbb{N}^+$ and m-cells σ in $\mathcal{C}$ bounded by 1-cells $\alpha, \beta : x \to y$.

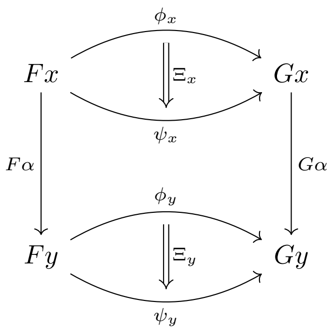
Definition 3.7 (internal ∞-natural higher transformations)**.** For each $n \in \mathbb{N}$ such that $n \geqslant 2$, an ***$\mathcal{M}$-internal ∞-natural n-transformation*** between $\mathcal{M}$-internal ∞-natural $(n-1)$-transformations ϕ and ψ both bounded by F and G is a morphism $\Xi : T.\mathcal{C}_0 \to T.\mathcal{D}_n$ in $\mathcal{M}$, written $\Xi : \phi \to \psi$, that satisfies

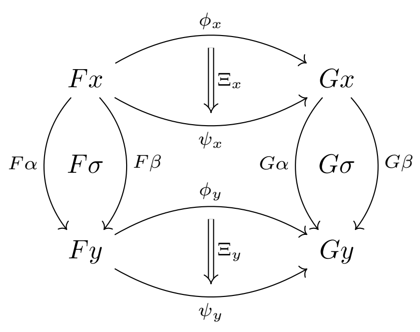
1. (Source and target) $\mathrm{s} \circ \Xi = \phi : T.\mathcal{C}_0 \to T.\mathcal{D}_{n-1}$ and $\mathrm{t} \circ \Xi = \psi : T.\mathcal{C}_0 \to T.\mathcal{D}_{n-1}$;
2. (Naturality) $*_0 \circ \langle \mathrm{i}^{n-1} \circ G_1, \Xi \circ \mathrm{s} \rangle_{\mathcal{D}_0} = *_0 \circ \langle \Xi \circ \mathrm{t}, \mathrm{i}^{n-1} \circ F_1 \rangle_{\mathcal{D}_0} : T.\mathcal{C}_1 \to T.\mathcal{D}_n$.

We also say that Ξ is ***bounded*** by F and G (n.b., $\mathrm{s}^n \circ \Xi = F_0$ and $\mathrm{t}^n \circ \Xi = G_0$).

Example 3.8. Set-internal ∞-natural n-transformations are exactly ∞-natural n-transformations [War11, §2.1], and in this sense $\mathcal{M}$-internal ∞-natural n-transformations generalise ∞-natural n-transformations.

When $\mathcal{M} = \mathrm{Set}$, the naturality of a Set-internal ∞-natural n-transformation $\Xi : \phi \to \psi$ bounded by Set-internal ∞-functors $F, G : \mathcal{C} \to \mathcal{D}$ corresponds to the commutativity of the left-lower diagram





in Set, where we omit identities i for brevity, and $\alpha : x \to y$ is an arbitrary 1-cell in $\mathcal{C}$.

A proper generalisation of the naturality of ∞-natural transformations is, however, the commutativity of the right-upper diagram, where we again omit identities i for brevity, and σ is an arbitrary m-cell in $\mathcal{C}$ ($m \in \mathbb{N}$) bounded by 1-cells $\alpha, \beta : x \to y$. In terms of the internal language of $\mathcal{M}$, this naturality is: $*_0 \circ \langle \mathrm{i}^{n-1-p} \circ G_{p+1}, \Xi \circ \mathrm{s}^{p+1} \rangle_{T.\mathcal{D}_0} = *_0 \circ \langle \Xi \circ \mathrm{t}^{p+1}, \mathrm{i}^{n-1-p} \circ F_{p+1} \rangle_{T.\mathcal{D}_0} : T.\mathcal{C}_{p+1} \to T.\mathcal{D}_n$ if $n > p-1$, and $*_0 \circ \langle G_{p+1}, \mathrm{i}^{p-(n-1)} \circ \Xi \circ \mathrm{s}^{p+1} \rangle_{T.\mathcal{D}_0} = *_0 \circ \langle \mathrm{i}^{p-(n-1)} \circ \Xi \circ \mathrm{t}^{p+1}, F_{p+1} \rangle_{T.\mathcal{D}_0} : T.\mathcal{C}_{p+1} \to T.\mathcal{D}_{p+1}$ otherwise.

By focusing on the case $m = 0$ in this generalised naturality, we recover the naturality of Ξ. Further, the naturality of Ξ is actually equivalent to the generalised naturality, which is why we choose the simpler one. This equivalence is a straightforward generalisation of [War11, Lemma 2.1], where the interchange and the unit laws of $\mathcal{M}$-internal ∞-categories play crucial roles; we leave the details to the reader.

The following is a straightforward generalisation of [War11, Proposition 2.2]:

Proposition 3.9 (∞-category of internal ∞-categories)**.** *The category $\infty\mathcal{M}\mathrm{Cat}$ of $\mathcal{M}$-internal ∞-categories and $\mathcal{M}$-internal ∞-functors gives rise to an ∞-category whose $(n+2)$-cells ($n \in \mathbb{N}$) are $\mathcal{M}$-internal ∞-natural $(n+1)$-transformations.*

Proof. First, the diagram for the globular set of $\infty\mathcal{M}\mathrm{Cat}$ (i.e., the sets of cells, the source function and the target function) is the evident one, so we leave the details to the reader.

Next, the identities $\mathrm{i}_n : \infty\mathcal{M}\mathrm{Cat}_n \to \infty\mathcal{M}\mathrm{Cat}_{n+1}$ ($n \in \mathbb{N}$) are the following functions:

- $\mathrm{i}_0 : \infty\mathcal{M}\mathrm{Cat}_0 \to \infty\mathcal{M}\mathrm{Cat}_1$ maps $\mathcal{M}$-internal ∞-categories $\mathcal{C}$ to the $\mathcal{M}$-internal ∞-functors $\mathrm{id}_{\mathcal{C}} := \lambda(\mathrm{v}) \in \mathrm{Tm}(T.N, \mathcal{C} \Rightarrow \mathcal{C}^+)$;
- $\mathrm{i}_1 : \infty\mathcal{M}\mathrm{Cat}_1 \to \infty\mathcal{M}\mathrm{Cat}_2$ maps $\mathcal{M}$-internal ∞-functors F to the $\mathcal{M}$-internal ∞-natural transformations $\mathrm{id}_F := \mathrm{i}_0 \circ F_0 : T.\mathcal{C}_0 \to T.\mathcal{D}_1$;
- $\mathrm{i}_n : \infty\mathcal{M}\mathrm{Cat}_n \to \infty\mathcal{M}\mathrm{Cat}_{n+1}$ ($n > 1$) maps $\mathcal{M}$-internal ∞-natural $(n-1)$-transformations Ξ to the $\mathcal{M}$-internal ∞-natural n-transformations $\mathrm{id}_\Xi := \mathrm{i}_{n-1} \circ \Xi : T.\mathcal{C}_0 \to T.\mathcal{D}_n$.

Let us next define the horizontal compositions $*_0^{[n]}$ ($n \in \mathbb{N}$) in $\infty\mathcal{M}\mathrm{Cat}$:

- The horizontal composition $G *_0^{[1]} F : \mathcal{C} \to \mathcal{E}$ of $\mathcal{M}$-internal ∞-functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ is the term
$$G *_0^{[1]} F := \lambda(\mathrm{App}(G\{\mathrm{p}\}, \lambda^{-1}(F))) \in \mathrm{Tm}(T.N, \mathcal{C} \Rightarrow \mathcal{E}^+);$$
- The horizontal composition $\beta *_0^{[n]} \alpha : G *_0^{[1]} F \Rightarrow G' *_0^{[1]} F' : \mathcal{C} \to \mathcal{E}$ of $\mathcal{M}$-internal ∞-natural $(n-1)$-transformations $\alpha : F \Rightarrow F' : \mathcal{C} \to \mathcal{D}$ and $\beta : G \Rightarrow G' : \mathcal{D} \to \mathcal{E}$ ($n > 1$) is the morphism
$$\beta *_0^{[n]} \alpha := *_0^{[n-1]} \circ \langle G'_{n-1} \circ \alpha, \beta \circ F_0 \rangle_{T.\mathcal{E}_1} : T.\mathcal{C}_0 \to T.\mathcal{E}_{n-1},$$
where $*_0^{[n-1]}$ on the right-hand side is the composition in $\mathcal{E}$.

Finally, let us define higher compositions $*_p^{[n]}$ in $\infty\mathcal{M}\mathrm{Cat}$ ($0 < p < n$). Given $*_p^{[n]}$-composable n-cells in $\infty\mathcal{M}\mathrm{Cat}$ (or $\mathcal{M}$-internal ∞-natural $(n-1)$-transformations) Ξ and Ξ' bounded by $\mathcal{M}$-internal ∞-functors $F, G : \mathcal{C} \to \mathcal{D}$, where the $*_p^{[n]}$-composability of Ξ and Ξ' means $\mathrm{s}^{n-p} \circ \Xi = \mathrm{t}^{n-p} \circ \Xi'$, the $*_p^{[n]}$-composition $\Xi' *_p^{[n]} \Xi$ is the morphism
$$\Xi' *_p^{[n]} \Xi := \left(T.\mathcal{C}_0 \overset{\langle \Xi, \Xi' \rangle_{T.\mathcal{D}_{p-1}}}{\to} T.\mathcal{D}_{n-1} \times_{T.\mathcal{D}_{p-1}} T.\mathcal{D}_{n-1} \overset{*_{p-1}^{[n-1]}}{\to} T.\mathcal{D}_{n-1} \right),$$

where $*^{[n-1]}_{p-1}$ on the right-hand side is the composition in $\mathcal{D}$. Again, it is easy to check that these higher compositions $*^{[n]}_{p}$ in $\infty\mathcal{M}\mathrm{Cat}$ satisfy the required axioms.

Finally, it is straightforward to verify that these structures are well-defined and satisfies the required axioms, so we leave it to the reader. $\square$

Let us turn to defining a forgetful ∞-functor $\mathcal{M}\infty\mathrm{Cat} \to \infty\mathrm{Cat}$ (Proposition 3.13), where $\infty\mathrm{Cat}$ is the (large) ∞-category of small ∞-categories [War11, Proposition 2.2]:

Definition 3.10 (underlying ∞-categories)**.** The ***underlying ∞-category*** of an $\mathcal{M}$-internal ∞-category $\mathcal{C}$ is the small (strict) ∞-category $|\mathcal{C}|$ defined as follows:

- The set $|\mathcal{C}|_n$ of n-cells ($n \in \mathbb{N}$) is the hom-set

$$|\mathcal{C}|_n := \mathcal{M}(T, T.\mathcal{C}_n);$$

- The source and the target functions $|\mathrm{s}|_n, |\mathrm{t}|_n : |\mathcal{C}|_{n+1} \to |\mathcal{C}|_n$ ($n \in \mathbb{N}$) are given by

$$|\mathrm{s}|_n(\sigma) := \mathrm{s}_n \circ \sigma \qquad\qquad |\mathrm{t}|_n(\sigma) := \mathrm{t}_n \circ \sigma$$

  for all $(n+1)$-cells $\sigma \in |\mathcal{C}|_{n+1}$;

- The identity function $|\mathrm{i}|_n : |\mathcal{C}|_n \to |\mathcal{C}|_{n+1}$ ($n \in \mathbb{N}$) is given by

$$|\mathrm{i}|_n(\tau) := \mathrm{i}_n \circ \tau$$

  for all n-cells $\tau \in |\mathcal{C}|_n$;

- The composition function $|*|^{[n]}_p : |\mathcal{C}|_n \times_{|\mathcal{C}|_p} |\mathcal{C}|_n \to |\mathcal{C}|_n$ for all $n, p \in \mathbb{N}$ such that $n > p$ is given by

$$|*|^{[n]}_p(\alpha, \beta) := *^{[n]}_p \circ \langle \alpha, \beta \rangle_{\mathcal{C}_p}$$

  for all n-cells $\alpha, \beta \in |\mathcal{C}|_n$ that satisfy $|\mathrm{s}|^{n-p}(\alpha) = |\mathrm{t}|^{n-p}(\beta)$, where $|\mathcal{C}|_n \times_{|\mathcal{C}|_p} |\mathcal{C}|_n$ is the canonical set-theoretic pullback

$$\begin{array}{ccc} |\mathcal{C}|_n \times_{|\mathcal{C}|_p} |\mathcal{C}|_n & \xrightarrow{\pi_2} & |\mathcal{C}|_n \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle |\mathrm{t}|^{n-p}} \\ |\mathcal{C}|_n & \xrightarrow[|\mathrm{s}|^{n-p}]{} & |\mathcal{C}|_p \end{array}$$

Definition 3.11 (underlying ∞-functors)**.** The ***underlying ∞-functor*** of an $\mathcal{M}$-internal ∞-functor $F : \mathcal{C} \to \mathcal{D}$ is the (strict) ∞-functor $|F| : |\mathcal{C}| \to |\mathcal{D}|$ whose component $|F|_n : |\mathcal{C}|_n \to |\mathcal{D}|_n$ ($n \in \mathbb{N}$) is the function defined by

$$|F|_n(\sigma) := F_n \circ \sigma \in |\mathcal{D}|_n \quad (\sigma \in |\mathcal{C}|_n).$$

Definition 3.12 (underlying ∞-natural transformations)**.** The ***underlying ∞-natural n-transformation*** of an $\mathcal{M}$-internal ∞-natural n-transformation Ξ, where $n > 0$ and $\Xi : \mathcal{C}_0 \to \mathcal{D}_n$ in $\mathcal{M}$, is the (strict) ∞-natural n-transformation $|\Xi| : |\mathcal{C}|_0 \to |\mathcal{D}|_n$ defined by

$$|\Xi|_\sigma := \Xi \circ \sigma \in |\mathcal{D}|_n \quad (\sigma \in |\mathcal{C}|_0).$$

Proposition 3.13 (forgetful ∞-functor)**.** *The operations $|\_|$ on $\mathcal{M}$-internal ∞-categories, $\mathcal{M}$-internal ∞-functors and $\mathcal{M}$-internal ∞-natural n-transformations ($n > 0$) assemble into an ∞-functor $|\_| : \infty\mathcal{M}\mathrm{Cat} \to \infty\mathrm{Cat}$.*

Proof. Straightforward and left to the reader. $\square$

3.2 Internal ∞-groupoids

In this section we introduce our notion of *internal ∞-groupoids*, which are internal ∞-categories whose nonzero cells are *weakly* invertible in the sense described below. The main idea for our computational interpretation of HoTT is to assign the structure of *half adjoint equivalence* [Uni13, Definition 4.2.1] to the cells of $\mathcal{M}$-internal ∞-categories, leading to our internal ∞-groupoids. This structure is in general not unique for each nonzero cell, so we include it as part of internal ∞-groupoids. The resulting weak invertibility is a radical departure from Warren [War11] as he employs the strict invertibility [War11, §4].

Let us illustrate the idea of our ∞-groupoids internal to the CwF $\mathcal{M}$ by taking $\mathcal{M} = \mathrm{Set}$ as follows. We first take an ∞-category $\mathcal{C}$ internal to Set, which is an ordinary (strict) ∞-category. We then assign a cell $\mathrm{inv}_n(\sigma) \in \mathcal{C}_n$ to each nonzero cell $\sigma \in \mathcal{C}_n$ $(n > 0)$ as the *(weak) inverse* of σ. Thus, if $\sigma : A \to B$, then $\mathrm{inv}_n(\sigma) : B \to A$, and there are invertible cells $\mathrm{inv}_n(\sigma) *_{n-1} \sigma \xrightarrow{\sim} \mathrm{i}(A)$ and $\sigma *_{n-1} \mathrm{inv}_n(\sigma) \xrightarrow{\sim} \mathrm{i}(B)$. For our interpretation of HoTT, we also require that this assignment inv_n preserves compositions and identities, where the preservation of a composition $\tau *_p \sigma$ $(n > p)$ means $\mathrm{inv}_n(\tau *_p \sigma) = \mathrm{inv}_n(\sigma) *_p \mathrm{inv}_n(\tau)$ if $p = n-1$, and $\mathrm{inv}_n(\tau *_p \sigma) = \mathrm{inv}_n(\tau) *_p \mathrm{inv}_n(\sigma)$ otherwise. Further, for our *computational* interpretation, we make a choice of the higher cells, a *retraction witness* $\mathrm{ret}_n(\sigma) : \mathrm{inv}_n(\sigma) *_{n-1} \sigma \xrightarrow{\sim} \mathrm{i}(A)$ and a *section witness* $\mathrm{sec}_n : \sigma *_{n-1} \mathrm{inv}_n(\sigma) \xrightarrow{\sim} \mathrm{i}(B)$. For our interpretation, they must preserve compositions and identities. The preservation of a composition $\tau *_p \sigma$ under ret_n means $\mathrm{ret}_n(\tau *_p \sigma) = \mathrm{ret}_n(\sigma) *_n \big(\mathrm{i} \circ \mathrm{inv}_n(\sigma) *_{n-1} \mathrm{ret}_n(\tau) *_{n-1} \mathrm{i}(\sigma)\big)$ if $p = n-1$, and $\mathrm{ret}_n(\tau *_p \sigma) = \mathrm{ret}_n(\tau) *_p \mathrm{ret}_n(\sigma)$ otherwise. The case for sec_n is similar. Finally, we assign a *triangle witness* $\mathrm{tri}_n(\sigma) : \mathrm{i}(\sigma) *_{n-1} \mathrm{ret}_n \xrightarrow{\sim} \mathrm{sec}_n *_{n-1} \mathrm{i}(\sigma)$ to σ, which corresponds to the last piece of half adjoint equivalence. Again, this assignment must preserve compositions and identities. The preservation of a composition $\tau *_p \sigma$ means $\mathrm{tri}_n(\tau *_p \sigma) = \big(\mathrm{i}^2(\tau) *_{n-1} \mathrm{tri}_n(\sigma)\big) *_n \big(\mathrm{tri}(\tau) *_{n-1} \mathrm{i}^2(\sigma)\big)$ if $p = n-1$, which is well-defined by the preservation of composition under ret_n and sec_n, and $\mathrm{tri}_n(\tau *_p \sigma) = \mathrm{tri}_n(\tau) *_p \mathrm{tri}_n(\sigma)$ otherwise.

We then generalise this idea on the CwF Set to the arbitrary one $\mathcal{M}$:

Definition 3.14 (internal ∞-groupoids)**.** An ***$\mathcal{M}$-internal ∞-groupoid*** is an $\mathcal{M}$-internal ∞-category $\mathcal{C}$ equipped with terms

$$\mathrm{inv} \in \mathrm{Tm}(T.N, \mathcal{C}\{\mathrm{succ}\} \Rightarrow \mathcal{C}\{\mathrm{succ}\}^+) \qquad \mathrm{ret} \in \mathrm{Tm}(T.N, \mathcal{C}\{\mathrm{succ}\} \Rightarrow \mathcal{C}\{\mathrm{succ}^2\}^+)$$

$$\mathrm{sec} \in \mathrm{Tm}(T.N, \mathcal{C}\{\mathrm{succ}\} \Rightarrow \mathcal{C}\{\mathrm{succ}^2\}^+) \qquad \mathrm{tri} \in \mathrm{Tm}(T.N, \mathcal{C}\{\mathrm{succ}\} \Rightarrow \mathcal{C}\{\mathrm{succ}^3\}^+)$$

in $\mathcal{M}$, called the ***(weak) inverse***, the ***retraction witness***, the ***section witness*** and the ***triangle witness*** on n-cells in $\mathcal{C}$, respectively, where we define morphisms

$$\mathrm{inv}_{n+1} := \mathrm{d}(\mathrm{inv}\{\langle !, \underline{n}\rangle\}) : T.\mathcal{C}_{n+1} \to T.\mathcal{C}_{n+1} \qquad \mathrm{ret}_{n+1} := \mathrm{d}(\mathrm{inv}\{\langle !, \underline{n}\rangle\}) : T.\mathcal{C}_{n+1} \to T.\mathcal{C}_{n+2}$$

$$\mathrm{sec}_{n+1} := \mathrm{d}(\mathrm{inv}\{\langle !, \underline{n}\rangle\}) : T.\mathcal{C}_{n+1} \to T.\mathcal{C}_{n+2} \qquad \mathrm{tri}_{n+1} := \mathrm{d}(\mathrm{inv}\{\langle !, \underline{n}\rangle\}) : T.\mathcal{C}_{n+1} \to T.\mathcal{C}_{n+3}$$

for all $n \in \mathbb{N}$ and often omit the subscript $(\_)_n$ on them, that satisfy

1. (Inverse) $\mathrm{s} \circ \mathrm{inv}_n = \mathrm{t}$, $\mathrm{t} \circ \mathrm{inv}_n = \mathrm{s}$ and $\mathrm{inv}_n \circ \mathrm{i} = \mathrm{i}$ for all $n \in \mathbb{N}^+$, and $\mathrm{inv}_n \circ *_p^{[n]} = \begin{cases} *_{n-1}^{[n]} \circ \langle \mathrm{inv}_n \circ \varpi_2, \mathrm{inv}_n \circ \varpi_1 \rangle_{T.\mathcal{C}_p} & \text{if } p = n-1 \\ *_p^{[n]} \circ (\mathrm{inv}_n \times_{T.\mathcal{C}_p} \mathrm{inv}_n) & \text{otherwise} \end{cases}$ for all $p \in \mathbb{N}$ such that $p < n$;

2. (Retraction) $\mathrm{s} \circ \mathrm{ret}_n = *_{n-1}^{[n]} \circ \langle \mathrm{inv}_n, \mathrm{id}_{T.\mathcal{C}_n} \rangle_{T.\mathcal{C}_{n-1}}$, $\mathrm{t} \circ \mathrm{ret}_n = \mathrm{i} \circ \mathrm{s}$ and $\mathrm{ret}_n \circ \mathrm{i} = \mathrm{i}^2$ for all $n \in \mathbb{N}^+$, and $\mathrm{ret}_n \circ *_p^{[n]} = \begin{cases} *_n^{[n+1]} \circ \langle \mathrm{ret}_n \circ \varpi_2, \mu_n \rangle_{T.\mathcal{C}_n} & \text{if } p = n-1 \\ *_p^{[n+1]} \circ (\mathrm{ret}_n \times_{T.\mathcal{C}_p} \mathrm{ret}_n) & \text{otherwise} \end{cases}$ for all $p \in \mathbb{N}$ such that $p < n$, where $\mu_n := *_{n-1}^{[n+1]} \circ \langle *_{n-1}^{[n+1]} \circ \langle \mathrm{i} \circ \mathrm{inv}_n \circ \varpi_2, \mathrm{ret}_n \circ \varpi_1 \rangle_{T.\mathcal{C}_{n-1}}, \mathrm{i} \circ \varpi_2 \rangle_{T.\mathcal{C}_{n-1}}$;

3. (Section) $\mathrm{s} \circ \mathrm{sec}_n = *_{n-1}^{[n]} \circ \langle \mathrm{id}_{T.\mathcal{C}_n}, \mathrm{inv}_n \rangle_{T.\mathcal{C}_{n-1}}$, $\mathrm{t} \circ \mathrm{sec}_n = \mathrm{i} \circ \mathrm{t}$ and $\mathrm{sec}_n \circ \mathrm{i} = \mathrm{i}^2$ for all $n \in \mathbb{N}^+$, and $\mathrm{sec}_n \circ *_p^{[n]} = \begin{cases} *_n^{[n+1]} \circ \langle \mathrm{sec}_n \circ \varpi_1, \nu_n \rangle_{T.\mathcal{C}_n} & \text{if } p = n-1 \\ *_p^{[n+1]} \circ (\mathrm{sec}_n \times_{T.\mathcal{C}_p} \mathrm{sec}_n) & \text{otherwise} \end{cases}$ for all $p \in \mathbb{N}$ such that $p < n$, where $\nu_n := *_{n-1}^{[n+1]} \circ \langle *_{n-1}^{[n+1]} \circ \langle \mathrm{i} \circ \varpi_1, \mathrm{sec}_n \circ \varpi_2 \rangle_{T.\mathcal{C}_{n-1}}, \mathrm{i} \circ \mathrm{inv}_n \circ \varpi_1 \rangle_{T.\mathcal{C}_{n-1}}$;

4. (Triangle) $\mathrm{s} \circ \mathrm{tri}_n = *_{n-1}^{[n+1]} \circ \langle \mathrm{i}, \mathrm{ret}_n \rangle_{T.\mathcal{C}_{n-1}}$, $\mathrm{t} \circ \mathrm{tri}_n = *_{n-1}^{[n+1]} \circ \langle \mathrm{sec}_n, \mathrm{i} \rangle_{T.\mathcal{C}_{n-1}}$ and $\mathrm{tri}_n \circ \mathrm{i} = \mathrm{i}^3$ for all $n \in \mathbb{N}^+$, and $\mathrm{tri}_n \circ *_p^{[n]} = \begin{cases} *_n^{[n+2]} \circ \langle \omega_n, \omega'_n \rangle_{T.\mathcal{C}_n} & \text{if } p = n-1 \\ *_p^{[n+2]} \circ (\mathrm{tri}_n \times_{T.\mathcal{C}_p} \mathrm{tri}_n) & \text{otherwise} \end{cases}$ for all $p \in \mathbb{N}$ such that $p < n$, where $\omega_n := *_{n-1}^{[n+2]} \circ \langle \mathrm{i}^2 \circ \varpi_1, \mathrm{tri}_n \circ \varpi_2 \rangle_{T.\mathcal{C}_{n-1}}$ and $\omega'_n := *_{n-1}^{[n+2]} \circ \langle \mathrm{tri}_n \circ \varpi_1, \mathrm{i}^2 \circ \varpi_2 \rangle_{T.\mathcal{C}_{n-1}}$.

Example 3.15. We simply call Set-internal ∞-groupoids ***(small) ∞-groupoids***.

Definition 3.16 (underlying ∞-groupoids)**.** The ***underlying ∞-groupoid*** of each $\mathcal{M}$-internal ∞-groupoid Γ is a small ∞-groupoid $|\Gamma|$ whose ∞-categorical structure is given as in Definition 3.16, and inverses and witnesses are similarly given by the Yoneda embedding.

Definition 3.17 (internal ∞-groupoid functors)**.** An ***$\mathcal{M}$-internal ∞-groupoid functor*** is an $\mathcal{M}$-internal ∞-functor $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{M}$-internal ∞-groupoids $\mathcal{C}$ and $\mathcal{D}$ that preserves the components of $\mathcal{M}$-internal ∞-groupoids, i.e., for each $n \in \mathbb{N}^+$ it satisfies

$$F_n \circ \mathrm{inv}_n = \mathrm{inv}_n \circ F_n : T.\mathcal{C}_n \to T.\mathcal{D}_n \qquad F_{n+1} \circ \mathrm{ret}_n = \mathrm{ret}_n \circ F_n : T.\mathcal{C}_n \to T.\mathcal{D}_{n+1}$$

$$F_{n+1} \circ \mathrm{sec}_n = \mathrm{sec}_n \circ F_n : T.\mathcal{C}_n \to T.\mathcal{D}_{n+1} \qquad F_{n+2} \circ \mathrm{tri}_n = \mathrm{tri}_n \circ F_n : T.\mathcal{C}_n \to T.\mathcal{D}_{n+2}.$$

The following lemma is one of the technical highlights of the present work, in which the weak inverses and the witnesses play crucial roles:

Lemma 3.18 (∞-groupoid of internal ∞-groupoids)**.** *$\mathcal{M}$-internal ∞-groupoids, $\mathcal{M}$-internal ∞-groupoid functors and $\mathcal{M}$-internal ∞-natural n-transformations ($n > 0$) assemble into an ∞-groupoid $\infty\mathcal{M}\mathrm{Gpd}$, for which the meta-theoretic ∞-categorical structure used below is that of $\infty\mathcal{M}\mathrm{Cat}$ (given in the proof of Proposition 3.9), such that*

- *0-cells are $\mathcal{M}$-internal ∞-groupoids;*
- *A 1-cell $\mathcal{C} \to \mathcal{D}$ is a quintuple $(F, G, \eta, \epsilon, \gamma)$ of $\mathcal{M}$-internal ∞-groupoid functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, $\mathcal{M}$-internal ∞-natural transformations $\eta : G *_0 F \to \mathrm{i}(\mathcal{C})$ and $\epsilon : F *_0 G \to \mathrm{i}(\mathcal{D})$, and an $\mathcal{M}$-internal ∞-natural 2-transformation $\gamma : \mathrm{i}(F) *_0 \eta \to \epsilon *_0 \mathrm{i}(F)$;*
- *If $n > 1$, then an n-cell $(\alpha, \beta, \eta, \epsilon, \gamma) \to (\alpha', \beta', \eta', \epsilon', \gamma')$ is a quintuple $(\phi, \psi, \sigma, \tau, \Xi)$ of $\mathcal{M}$-internal ∞-natural $(n-1)$-transformations $\phi : \alpha \rightleftarrows \alpha' : \psi$, $\mathcal{M}$-internal ∞-natural n-transformations $\sigma : \psi *_{n+1} \phi \to \mathrm{i}(\alpha)$ and $\tau : \phi *_{n+1} \psi \to \mathrm{i}(\alpha')$, and an $\mathcal{M}$-internal ∞-natural $(n+1)$-transformation $\Xi : \mathrm{i}(\phi) *_{n+1} \sigma \to \tau *_{n+1} \mathrm{i}(\phi)$;*
- *The source and the target of each nonzero cell have been already defined;*
- *The identity on each 0-cell $\mathcal{C}$ is the quintuple*

$$\mathrm{i}_0(\mathcal{C}) := \big(\mathrm{i}_0(\mathcal{C}), \mathrm{i}_0(\mathcal{C}), \mathrm{i}_1 \circ \mathrm{i}_0(\mathcal{C}), \mathrm{i}_1 \circ \mathrm{i}_0(\mathcal{C}), \mathrm{i}_2 \circ \mathrm{i}_1 \circ \mathrm{i}_0(\mathcal{C})\big);$$

- *For each $n \in \mathbb{N}$, the identity on each $(n+1)$-cell $(\alpha, \beta, \sigma, \tau, \Xi)$ is the quintuple*

$$\mathrm{i}_{n+1}(\alpha, \beta, \sigma, \tau, \Xi) := \big(\mathrm{i}_{n+1}(\alpha), \mathrm{i}_{n+1}(\alpha), \mathrm{i}_{n+2} \circ \mathrm{i}_{n+1}(\alpha), \mathrm{i}_{n+2} \circ \mathrm{i}_{n+1}(\alpha), \mathrm{i}_{n+3} \circ \mathrm{i}_{n+2} \circ \mathrm{i}_{n+1}(\alpha)\big);$$

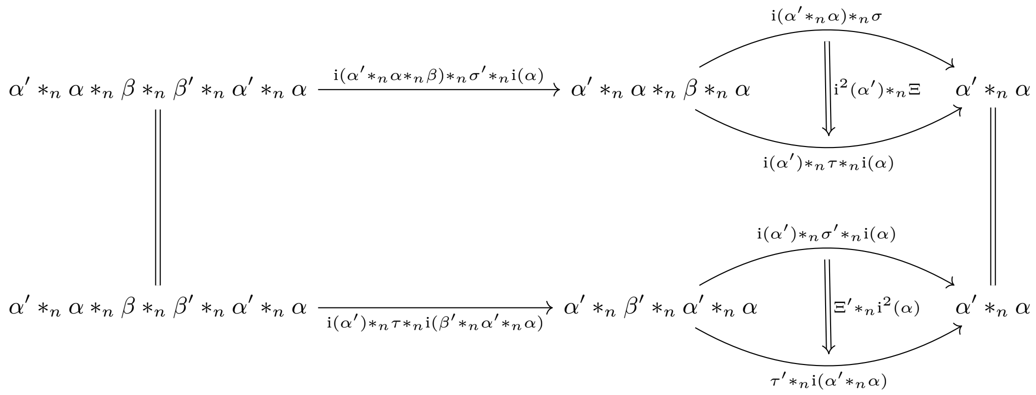
- *The $*_p$-composition of n-cells $(\alpha, \beta, \sigma, \tau, \Xi)$ and $(\alpha', \beta', \sigma', \tau', \Xi')$, where $p, n \in \mathbb{N}$ and $p < n$, such that $\mathrm{s}^{n-p}(\alpha', \beta', \sigma', \tau', \Xi') = \mathrm{t}^{n-p}(\alpha, \beta, \sigma, \tau, \Xi)$ is the quintuple*

$$\begin{aligned} &(\alpha', \beta', \sigma', \tau', \Xi') *_p (\alpha, \beta, \sigma, \tau, \Xi) \\ :=& \begin{cases} \big(\alpha' *_n \alpha, \beta *_n \beta', \sigma *_{n+1} (\mathrm{i}(\beta) *_n \sigma' *_n \mathrm{i}(\alpha)), \tau' *_{n+1} (\mathrm{i}(\alpha') *_n \tau *_n \mathrm{i}(\beta')), \mathrm{com}_n(\Xi, \Xi')\big) & \textit{if } p = n-1 \\ \big(\alpha' *_p \alpha, \beta' *_p \beta, \sigma' *_p \sigma, \tau' *_p \tau, \Xi' *_p \Xi\big) & \textit{otherwise,} \end{cases} \end{aligned}$$

where $\mathrm{com}_n(\Xi, \Xi')$ is the $\mathcal{M}$-internal ∞-natural $(n+2)$-transformation

$$\begin{aligned} &\mathrm{com}_n(\Xi, \Xi') \\ :=& \Big(\big(\Xi' *_n \mathrm{i}^2(\alpha)\big) *_{n+1} \mathrm{i}\big(\mathrm{i}(\alpha') *_n \tau *_n \mathrm{i}(\beta' *_n \alpha' *_n \alpha)\big)\Big) *_{n+2} \Big(\big(\mathrm{i}^2(\alpha') *_n \Xi\big) *_{n+1} \mathrm{i}\big(\mathrm{i}(\alpha' *_n \alpha *_n \beta) *_n \sigma' *_n \mathrm{i}(\alpha)\big)\Big); \end{aligned}$$

Remark. Schematically, $\mathrm{com}_n(\Xi, \Xi')$ is the $(n+3)$-cell in $\infty\mathcal{M}\mathrm{Cat}$ displayed vertically in the diagram



where the $(n+2)$-composition makes sense because

$$\begin{aligned}\big(\mathrm{i}(\alpha') *_n \sigma' *_n \mathrm{i}(\alpha)\big) *_{n+1} \big(\mathrm{i}(\alpha') *_n \tau *_n \mathrm{i}(\beta') *_n \mathrm{i}(\alpha' *_n \alpha)\big) &= \mathrm{i}(\alpha') *_n \Big(\sigma' *_{n+1} \big(\tau *_n \mathrm{i}(\beta' *_n \alpha')\big)\Big) *_n \mathrm{i}(\alpha) \\ &= \mathrm{i}(\alpha') *_n \Big(\tau *_{n+1} \big(\mathrm{i}(\alpha *_n \beta) *_n \sigma'\big)\Big) *_n \mathrm{i}(\alpha) \\ &= \big(\mathrm{i}(\alpha') *_n \tau *_n \mathrm{i}(\alpha)\big) *_{n+1} \big(\mathrm{i}(\alpha' *_n \alpha *_n \beta) *_n \sigma' *_n \mathrm{i}(\alpha)\big).\end{aligned}$$

- *For each $n \in \mathbb{N}$, the inverse of each $(n+1)$-cell $(\alpha, \beta, \sigma, \tau, \Xi)$ is the quadruple*

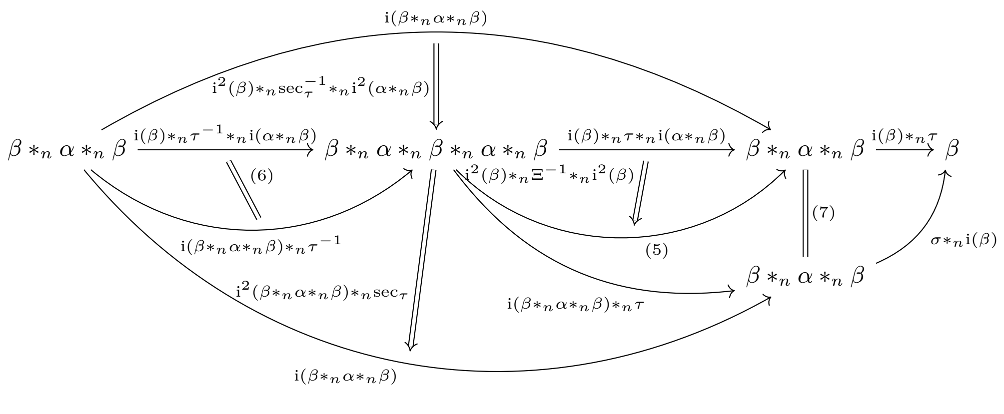
$$\mathrm{inv}_{n+1}(\alpha, \beta, \sigma, \tau, \Xi) := (\beta, \alpha, \tau, \sigma, \mathrm{dual}_n(\Xi)),$$

where $\mathrm{dual}_n(\Xi)$ *is the $\mathcal{M}$-internal ∞-natural $(n+2)$-transformation*

$$\begin{aligned}\mathrm{dual}_n(\Xi) := &\Big(\mathrm{i}\big(\sigma *_n \mathrm{i}(\beta)\big) *_{n+1} \big(\mathrm{i}^2(\beta *_n \alpha *_n \beta) *_n \sec_\tau\big)\Big) \\ &*_{n+2} \Big(\mathrm{i}\big(\mathrm{i}(\beta) *_n \tau\big) *_{n+1} \big(\mathrm{i}^2(\beta) *_n \Xi^{-1} *_n \mathrm{i}^2(\beta)\big) *_{n+1} \mathrm{i}\big(\mathrm{i}(\beta *_n \alpha *_n \beta) *_n \tau^{-1}\big)\Big) \\ &*_{n+2} \Big(\mathrm{i}\big(\mathrm{i}(\beta) *_n \tau\big) *_{n+1} \big(\mathrm{i}^2(\beta) *_n \sec_\tau^{-1} *_n \mathrm{i}^2(\alpha *_n \beta)\big)\Big),\end{aligned}$$

and $\tau^{-1} := \mathrm{inv} \circ \tau$, $\Xi^{-1} := \mathrm{inv} \circ \Xi$, $\sec_\tau := \sec \circ \tau$ *and* $\sec_\tau^{-1} := \mathrm{inv} \circ \sec \circ \tau$.

Remark. Schematically, $\mathrm{dual}_n(\Xi)$ is the $(n+3)$-cell in $\infty\mathcal{M}\mathrm{Cat}$ displayed vertically in the diagram



where $\mathrm{dual}_n(\Xi)$ is well-defined since

$$\mathrm{i}(\beta *_n \alpha) *_n \sigma *_n \mathrm{i}(\beta) = \sigma *_n \mathrm{i}(\beta *_n \alpha *_n \beta) \tag{5}$$

holds by the naturality of σ,

$$\mathrm{i}(\beta) *_n \tau^{-1} *_n \mathrm{i}(\alpha *_n \beta) = \mathrm{i}(\beta *_n \alpha *_n \beta) *_n \tau^{-1} \tag{6}$$

holds by the naturality of τ and the functoriality of inv, and

$$\begin{aligned}\big(\mathrm{i}(\beta) *_n \tau\big) *_{n+1} \big(\sigma *_n \mathrm{i}(\beta *_n \alpha *_n \beta)\big) &= \sigma *_n \mathrm{i}(\beta) *_n \tau \\ &= \big(\sigma *_n \mathrm{i}(\beta)\big) *_{n+1} \big(\mathrm{i}(\beta *_n \alpha *_n \beta) *_n \tau\big) \end{aligned} \tag{7}$$

holds by the interchange laws.

- *The retraction witness of each $(n+1)$-cell $(\alpha, \beta, \sigma, \tau, \Xi)$ is the quadruple*

$$\mathrm{ret}_{n+1}(\alpha, \beta, \sigma, \tau, \Xi) := (\sigma, \mathrm{inv} \circ \sigma, \mathrm{ret} \circ \sigma, \mathrm{sec} \circ \sigma, \mathrm{tri} \circ \sigma);$$

- *The section witness of each $(n+1)$-cell $(\alpha, \beta, \sigma, \tau, \Xi)$ is the quadruple*

$$\mathrm{sec}_{n+1}(\alpha, \beta, \sigma, \tau, \Xi) := (\tau, \mathrm{inv} \circ \tau, \mathrm{ret} \circ \tau, \mathrm{sec} \circ \tau, \mathrm{tri} \circ \tau);$$

- *The triangle witness of an $(n+1)$-cell $(\alpha, \beta, \sigma, \tau, \Xi)$ is the quintuple*

$$\mathrm{tri}_{n+1}(\alpha, \beta, \sigma, \tau, \Xi) := (\Xi, \mathrm{inv} \circ \Xi, \mathrm{ret} \circ \Xi, \mathrm{sec} \circ \Xi, \mathrm{tri} \circ \Xi).$$

Proof. Straightforward yet lengthly. □

Notation. Let us define $\infty\mathrm{Gpd} := \infty\mathrm{SetGpd}$.

It is evident how to lift the forgetful ∞-functor $|\_| : \infty\mathcal{M}\mathrm{Cat} \to \infty\mathrm{Cat}$ (Proposition 3.13) into a forgetful ∞-groupoid functor $|\_| : \infty\mathcal{M}\mathrm{Gpd} \to \infty\mathrm{Gpd}$:

Proposition 3.19 (forgetful ∞-groupoid functor)**.** *There is an ∞-groupoid functor $|\_| : \infty\mathcal{M}\mathrm{Gpd} \to \infty\mathrm{Gpd}$ that maps $\mathcal{M}$-internal ∞-groupoids Γ into their underlying ∞-groupoids $|\Gamma|$ (Definition 3.16), $\mathcal{M}$-internal ∞-groupoid functors F into their underlying ∞-functors (Definition 3.11), and $\mathcal{M}$-internal ∞-natural n-transformations Ξ ($n > 0$) into their underlying ∞-natural n-transformations (Definition 3.12).*

3.3 Internal universes and internal dependent ∞-groupoids

By Lemma 3.18, it seems reasonable to follow Warren [War11] in his method to interpret types over a context by ∞-groupoid functors $|\Gamma| \to \mathcal{M}\infty\mathrm{Gpd}$, where Γ is the $\mathcal{M}$-internal ∞-groupoid that interprets the context. For our *computational* interpretation of HoTT, however, we have to *encode* these ∞-groupoid functors $|\Gamma| \to \mathcal{M}\infty\mathrm{Gpd}$ by $\mathcal{M}$-internal ∞-functors $A : \Gamma \to \tilde{\mathcal{U}}_k^\infty$, which we call *internal dependent ∞-groupoids* over Γ, for some sufficiently large index $k \in \mathbb{N}$, where the $\mathcal{M}$-internal ∞-groupoid $\tilde{\mathcal{U}}_k^\infty$ is obtained from the corresponding universe $\mathcal{U}_k$ in $\mathcal{M}$. The aim of this section is to construct these internal universes $\tilde{\mathcal{U}}_k^\infty$ and internal dependent ∞-groupoids A. Parallel to the construction of the ∞-groupoid $\infty\mathrm{Gpd}$ through the ∞-category $\infty\mathrm{Cat}$, we construct $\tilde{\mathcal{U}}_k^\infty$ through more primitive internal universes $\mathcal{U}_k^\infty$.

Notation. Let $\Gamma = T.A_1.A_2 \dots A_n \in \mathcal{M}$ be an object with the length $\mathrm{lth}(\Gamma) = n$. For each $i \in \{1, 2, \dots, n\}$, we write $\pi_i^{[n]} \in \mathrm{Tm}(\Gamma, A_i\{\mathrm{p}^{n-i+1}\})$ for the term defined by

$$\pi_i^{[n]} := \mathrm{v}\{\mathrm{p}^{n-i}\}.$$

In the following, we often internalise the composition of morphisms in $\mathcal{M}$ to itself:

Lemma 3.20 (internal composition)**.** *Given objects $\Delta, \Gamma, \Theta \in \mathcal{M}$, there is a morphism $(\Gamma \Rightarrow \Theta) \times (\Delta \Rightarrow \Gamma) \to (\Delta \Rightarrow \Theta)$ in $\mathcal{M}$ that internalises the composition $\mathcal{M}(\Gamma, \Theta) \times \mathcal{M}(\Delta, \Gamma) \to \mathcal{M}(\Delta, \Theta)$ of morphisms in $\mathcal{M}$.*

Proof. By the democracy and the Pi-types of $\mathcal{M}$. □

We begin with defining the internal universes $\mathcal{U}_k^\infty$. Let us fix $k \in \mathbb{N}$. As a preparation, we first define an object $\mathcal{U}_{k,0}^\infty \in \mathcal{M}$ for encoding $\mathcal{M}$-internal categories by

$$\mathcal{U}_{k,0}^\infty := T.\Pi(N, \mathcal{U}_k).\Pi(N, \mathrm{S}).\Pi(N, \mathrm{T}).\Pi(N.N_b, \mathrm{Comp}).\Pi(N, \mathrm{I}).\mathrm{InfCatAx},$$

where

$$\mathrm{S} := \mathrm{El}\{\mathrm{App}(\pi_1^{[2]}, \pi_2^{[2]})\}^{\mathrm{El}\{\mathrm{App}(\pi_1^{[2]}, \pi_2^{[2]}\{\mathrm{succ}\})\}} \in \mathrm{Ty}(T.\Pi(N, \mathcal{U}_k).N)$$
$$\mathrm{T} := \mathrm{El}\{\mathrm{App}(\pi_1^{[3]}, \pi_3^{[3]})\}^{\mathrm{El}\{\mathrm{App}(\pi_1^{[3]}, \pi_3^{[3]}\{\mathrm{succ}\})\}} \in \mathrm{Ty}(T.\Pi(N, \mathcal{U}_k).\Pi(N, \mathrm{S}).N)$$
$$\mathrm{I} := \mathrm{El}\{\mathrm{App}(\pi_1^{[5]}, \pi_5^{[5]}\{\mathrm{succ}\})\}^{\mathrm{El}\{\mathrm{App}(\pi_1^{[5]}, \pi_5^{[5]})\}} \in \mathrm{Ty}(T.\Pi(N, \mathcal{U}_k).\Pi(N, \mathrm{S}).\Pi(N, \mathrm{T}).\Pi(N.N_b, \mathrm{Comp}).N),$$

and InfCatAx is an internalisation of the axioms of $\mathcal{M}$-internal categories that admits only the trivial term o (due to the extensionality of the Id-types pId). It is evident how to implement InfCatAx, and the details are unimportant. Hence, we skip presenting the precise definition of InfCatAx (just like skipping a complete formalisation of mathematics by an axiomatic set theory). Each global element $\mu \in |\mathcal{U}_{k,0}^{\infty}|$ encodes an $\mathcal{M}$-internal ∞-category $\mathrm{El}(\mu)$, where the first nontrivial component $\Pi(N, \mathcal{U}_k)$ of $\mathcal{U}_{k,0}^{\infty}$ specifies the type $\mathrm{El}(\mu) \in \mathrm{Ty}(N)$, the second one $\Pi(N, \mathrm{S})$ does the sources, the third one $\Pi(N, \mathrm{T})$ does the targets, the fourth one $\Pi(N.N_b, \mathrm{Comp})$ does the compositions, and the fifth one $\Pi(N, \mathrm{I})$ does the identities.

Next, we define an object $\mathcal{U}_{k,1}^{\infty} \in \mathcal{M}$ for encoding $\mathcal{M}$-internal ∞-functors by

$$\mathcal{U}_{k,1}^{\infty} := T.\mathrm{d}(\mathcal{U}_{k,0}^{\infty}).\mathrm{d}(\mathcal{U}_{k,0}^{\infty})\{\mathrm{p}\}.\Pi(N, \mathrm{InfFunTerm}).\mathrm{InfFunAx},$$

where

$$\mathrm{InfFunTerm} := \mathrm{El}\{\mathrm{App}(\pi_1^{[6]}\{\langle !, \pi_2^{[3]}\rangle\}, \pi_3^{[3]})\}^{\mathrm{El}\{\mathrm{App}(\pi_1^{[6]}\{\langle !, \pi_1^{[3]}\rangle\}, \pi_3^{[3]})\}} \in \mathrm{Ty}(T.\mathrm{d}(\mathcal{U}_{k,0}^{\infty}).\mathrm{d}(\mathcal{U}_{k,0}^{\infty})\{\mathrm{p}\}.N),$$

and InfFunAx is an internalisation of the axioms of $\mathcal{M}$-internal ∞-functors that admits only the trivial term o (due to the extensionality of pId). Similarly to the case of InfCatAx, we skip presenting the precise definition of InfFunAx. Each global element $\phi \in |\mathcal{U}_{k,1}^{\infty}|$ encodes an $\mathcal{M}$-internal ∞-groupoid functor $\mathrm{El}(\phi)$, where the first nontrivial component $\mathrm{d}(\mathcal{U}_{k,0}^{\infty})$ of $\mathcal{U}_{k,1}^{\infty}$ specifies the domain of $\mathrm{El}(\phi)$, the second one $\mathrm{d}(\mathcal{U}_{k,0}^{\infty})\{\mathrm{p}\}$ does the codomain of $\mathrm{El}(\phi)$, and the third one $\Pi(N, \mathrm{InfFunTerm})$ does the term of $\mathrm{El}(\phi)$.

Finally, for each $n \geqslant 1$, we define an object $\mathcal{U}_{k,n}^{\infty} \in \mathcal{M}$ for internalising $\mathcal{M}$-internal ∞-natural n-transformations by

$$\mathcal{U}_{k,n+1}^{\infty} := T.\mathrm{d}(\mathcal{U}_{k,n}^{\infty}).\mathrm{d}(\mathcal{U}_{k,n}^{\infty})\{\mathrm{p}\}.n\mathrm{InfTraTerm}.n\mathrm{InfTraAx},$$

where

$$n\mathrm{InfTraTerm} := \mathrm{El}\{\mathrm{App}(\pi_1^{[4]}\{\langle !, \pi_1^{[2]}\rangle\}, \underline{n})\}^{\mathrm{El}\{\mathrm{App}(\pi_1^{[4]}\{\langle !, \pi_1^{[2]}\rangle\}, \underline{0})\}} \in \mathrm{Ty}(T.\mathrm{d}(\mathcal{U}_{k,n}^{\infty}).\mathrm{d}(\mathcal{U}_{k,n}^{\infty})\{\mathrm{p}\}),$$

and InfTraAx is an internalisation of the axioms of $\mathcal{M}$-internal ∞-natural n-transformations that admits only the trivial term o (due to the extensionality of pId). Again, we skip presenting the precise definition of InfTraAx. Each global element $\xi \in |\mathcal{U}_{k,n+1}^{\infty}|$ encodes an $\mathcal{M}$-internal ∞-natural n-transformation $\mathrm{El}(\xi)$, where the first nontrivial component $\mathrm{d}(\mathcal{U}_{k,n}^{\infty})$ of $\mathcal{U}_{k,n+1}^{\infty}$ specifies the domain of α, the second one $\mathrm{d}(\mathcal{U}_{k,n}^{\infty})\{\mathrm{p}\}$ does the codomain of $\mathrm{El}(\xi)$, and the third one nInfTraTerm does the term of $\mathrm{El}(\xi)$.

We are now ready to define the internal universe $\mathcal{U}_k^{\infty}$:

Definition 3.21 (internal ∞-universes for internal ∞-categories)**.** The ***$\mathcal{M}$-internal $(k+1)$st ∞-universe*** induced by the universe $\mathcal{U}_k$ in $\mathcal{M}$ ($k \in \mathbb{N}$) is the $\mathcal{M}$-internal ∞-groupoid $\mathcal{U}_k^{\infty}$ such that

- The type $\mathcal{U}_k^{\infty} \in \mathrm{Ty}(T.N)$ is obtained from the family $(\mathcal{U}_{k,n}^{\infty})_{n \in \mathbb{N}}$ by the elimination rule of N-type with respect to the $(k+2)$nd universe $\mathcal{U}_{k+1}$ in $\mathcal{M}$ (n.b., this is why we need not only a single universe but also an infinite hierarchy of universes in $\mathcal{M}$), in the evident way that formalises the inductive definition of the objects $\mathcal{U}_{k,n}^{\infty}$ given above (n.b., again, the precise definition of $\mathcal{U}_k^{\infty}$ is quite tedious yet unimportant, so we skip presenting it), where note that the notation $\mathcal{U}_{k,n}^{\infty}$ is unambiguous;
- The sources and the targets $\mathrm{s}, \mathrm{t} \in \mathrm{Tm}(T.N, \mathcal{U}_k^{\infty}\{\mathrm{succ}\}, (\mathcal{U}_k^{\infty})^+)$ are given by the evident projections;
- For each $n, p \in \mathbb{N}$ with $n > p$, the composition $*_p^{[n]} : \mathcal{U}_{k,n}^{\infty} \times_{\mathcal{U}_{k,p}^{\infty}} \mathcal{U}_{k,n}^{\infty} \to \mathcal{U}_{k,n}^{\infty}$ is the evident internalisation of the corresponding composition in $\mathcal{M}\infty\mathrm{Cat}$;
- For each $n \in \mathbb{N}$, the identity $\mathrm{i}_n : \mathcal{U}_{k,n}^{\infty} \to \mathcal{U}_{k,n+1}^{\infty}$ is the evident internalisations of the corresponding identity in $\mathcal{M}\infty\mathrm{Cat}$.

Lemma 3.22. *The structure $\mathcal{U}_k^\infty$ forms a well-defined $\mathcal{M}$-internal ∞-category for all $k \in \mathbb{N}$.*

Proof. The lemma holds because $\mathcal{U}_k^\infty$ internalises the required axioms. □

Let us write $\infty\mathcal{M}_\mathcal{U}\mathrm{Cat} \hookrightarrow \infty\mathcal{M}\mathrm{Cat}$ for the ∞-category whose 0-cells are encodable by morphisms $T \to \mathcal{U}_k$ in $\mathcal{M}$ for some index $k \in \mathbb{N}$. Similarly, let us write $\infty\mathcal{M}_\mathcal{U}\mathrm{Gpd} \hookrightarrow \infty\mathcal{M}\mathrm{Gpd}$ for the ∞-groupoid whose 0-cells are encodable by morphisms $T \to \mathcal{U}_k$ in $\mathcal{M}$ for some $k \in \mathbb{N}$. It is then clear that the $\mathcal{M}$-internal ∞-categories $\mathcal{U}_k^\infty$ collectively internalise the ∞-category $\infty\mathcal{M}_\mathcal{U}\mathrm{Cat}$: We can lift the elimination constructor El of the universe $\mathcal{U}_k$ in $\mathcal{M}$ to a surjection $\mathrm{El} : |\mathcal{U}_k^\infty| \twoheadrightarrow \infty\mathcal{M}\mathrm{Cat}$ in the evident way.

Next, by internalising the construction given in Lemma 3.18, we lift the $\mathcal{M}$-internal ∞-categories $\mathcal{U}_k^\infty \in \infty\mathcal{M}\mathrm{Cat}$ for all $k \in \mathbb{N}$ to $\mathcal{M}$-internal ∞-groupoids $\tilde{\mathcal{U}}_k^\infty \in \infty\mathcal{M}\mathrm{Gpd}$ in such a way that they collectively internalise the ∞-groupoid $\infty\mathcal{M}_\mathcal{U}\mathrm{Gpd}$. For instance, similarly to the family $(\mathcal{U}_{k,n}^\infty)_{n\in\mathbb{N}}$ of objects $\mathcal{U}_{k,n}^\infty \in \mathcal{M}$, we define a family $(\tilde{\mathcal{U}}_{k,n}^\infty)_{n\in\mathbb{N}}$ of objects $\tilde{\mathcal{U}}_{k,n}^\infty \in \mathcal{M}$ by

$$\tilde{\mathcal{U}}_{k,0}^\infty := T.\Pi(N,\mathcal{U}_k).\Pi(N,\mathrm{S}).\Pi(N,\mathrm{T}).\Pi(N.N_b,\mathrm{Comp}).\Pi(N,\mathrm{I}).\Pi(N,\mathrm{Inv}).\Pi(N,\mathrm{Ret}).\Pi(N,\mathrm{Sec}).\Pi(N,\mathrm{Tri}).\mathrm{InfGpdAx},$$

where

$$\begin{aligned}
\mathrm{Inv} &:= \mathrm{El}\{\mathrm{App}(\pi_1^{[6]},\pi_4^{[6]})\}^{\mathrm{El}\{\mathrm{App}(\pi_1^{[6]},\pi_4^{[6]})\}} \\
\mathrm{Ret} &:= \mathrm{El}\{\mathrm{App}(\pi_1^{[7]},\pi_4^{[7]}\{\mathrm{succ}\})\}^{\mathrm{El}\{\mathrm{App}(\pi_1^{[7]},\pi_4^{[7]})\}} \\
\mathrm{Sec} &:= \mathrm{El}\{\mathrm{App}(\pi_1^{[8]},\pi_4^{[8]}\{\mathrm{succ}\})\}^{\mathrm{El}\{\mathrm{App}(\pi_1^{[8]},\pi_4^{[8]})\}} \\
\mathrm{Tri} &:= \mathrm{El}\{\mathrm{App}(\pi_1^{[9]},\pi_4^{[9]}\{\mathrm{succ}^2\})\}^{\mathrm{El}\{\mathrm{App}(\pi_1^{[9]},\pi_4^{[9]})\}},
\end{aligned}$$

and InfGpdAx is an internalisation of the axioms of $\mathcal{M}$-internal groupoids that admits only the trivial term o (due to the extensionality of pId). The additional components $\Pi(N,\mathrm{Inv})$, $\Pi(N,\mathrm{Ret})$, $\Pi(N,\mathrm{Sec})$ and $\Pi(N,\mathrm{Tri})$ respectively internalise the inverses and the witnesses.

At this point, having described the construction $\infty\mathcal{M}\mathrm{Cat} \mapsto \infty\mathcal{M}\mathrm{Gpd}$ (the proof of Lemma 3.18), the $\mathcal{M}$-internal universe $\mathcal{U}_k^\infty$ (Definition 3.21) and the objects $\tilde{\mathcal{U}}_{k,n}^\infty$ ($n \in \mathbb{N}$) as above, it is just a routine to internalise the construction $\infty\mathcal{M}\mathrm{Cat} \mapsto \infty\mathcal{M}\mathrm{Gpd}$ into a one $\mathcal{U}_k^\infty \mapsto \tilde{\mathcal{U}}_k^\infty$, obtaining the $\mathcal{M}$-internal ∞-groupoid $\tilde{\mathcal{U}}_k^\infty$. Moreover, it is straightforward to lift the elimination constructor El of $\mathcal{U}_k^\infty$ to $\tilde{\mathcal{U}}_k^\infty$.

Again, the details are tedious yet unimportant, so we just summarise the argument by:

Theorem 3.23 (internal universes for internal ∞-groupoids)**.** *There is an $\mathcal{M}$-internal ∞-groupoid $\tilde{\mathcal{U}}_k^\infty \in \infty\mathcal{M}\mathrm{Gpd}$ for each $k \in \mathbb{N}$, which we call the* **$\mathcal{M}$-internal $(k+1)$st ∞-groupoid universe***, such that there is a map $\mathrm{El} : |\tilde{\mathcal{U}}_k^\infty| \to \infty\mathcal{M}\mathrm{Gpd}$ compatible with the elimination rule of the universe $\mathcal{U}_k$ in $\mathcal{M}$.*

Convention. When the index k for the $(k+1)$st ∞-groupoid universe $\tilde{\mathcal{U}}_k^\infty$ is unimportant, we simply omit k and call $\tilde{\mathcal{U}}^\infty$ an ***$\mathcal{M}$-internal ∞-groupoid universe***.

By this theorem, the following makes sense:

Definition 3.24 (internal dependent ∞-groupoids)**.** An ***$\mathcal{M}$-internal dependent ∞-groupoid*** over an $\mathcal{M}$-internal ∞-groupoid $\Gamma \in \infty\mathcal{M}\mathrm{Gpd}$ is an $\mathcal{M}$-internal ∞-functor $\Gamma \to \tilde{\mathcal{U}}^\infty$.

Notation. We write $|\_| : \infty\mathcal{M}\mathrm{Gpd}(\Gamma,\tilde{\mathcal{U}}^\infty) \to \infty\mathrm{Gpd}(|\Gamma|,\infty\mathcal{M}\mathrm{Gpd})$ for the map that *decodes* $\mathcal{M}$-internal dependent ∞-groupoids into the corresponding ∞-groupoid functors.

Definition 3.25 (small internal ∞-grouopids)**.** We write $\infty\mathcal{M}_\mathcal{U}\mathrm{Gpd} \hookrightarrow \infty\mathcal{M}\mathrm{Gpd}$ for the ∞-groupoid whose 0-cells are encodable by morphisms $T \to \mathcal{U}_k$ in $\mathcal{M}$ for some $k \in \mathbb{N}$, which we call ***small***.

3.4 Internal Grothendieck construction

We next internalise Warren's ∞-groupoid interpretation of comprehension $\_.\_$ [War11, §3], which is an infinite-dimensional variant of *Grothendieck construction* [Bor94, §8.3]. For this task, it is crucial that we encode ∞-groupoid functors $|\Gamma| \to \infty\mathcal{M}_\mathcal{U}\mathrm{Gpd}$ for all $\Gamma \in \infty\mathcal{M}_\mathcal{U}\mathrm{Gpd}$ by $\mathcal{M}$-internal dependent ∞-groupoids $\Gamma \to \tilde{\mathcal{U}}^\infty$ (§3.3) as we shall see.

Notation. Before going into details, let us fix some notations for convenience. First, lifting the notation for $\mathcal{M}$-internal ∞-categories introduced in §3.1, we write $G[\sigma,\tau]$, where $G \in \infty\mathcal{M}\mathrm{Gpd}$ and $\sigma,\tau \in |G|_n$ for some $n \in \mathbb{N}$, for the substructural $\mathcal{M}$-internal ∞-groupoid of G bounded by σ and τ.

Next, given a small $\mathcal{M}$-internal ∞-groupoid $\Gamma \in \infty\mathcal{M}_{\mathcal{U}}\mathrm{Gpd}$, we define a type

$$\Gamma^{\star}_{\_}[\_,\_]_{\_} \in \mathrm{Ty}(T.N.\Gamma.\Gamma^{+}.N)$$

in $\mathcal{M}$ that satisfies the equation

$$\Gamma^{\star}_{\_}[\_,\_]_{\_}\{\langle !, \underline{n}, \gamma, \gamma', \underline{m} \rangle\} = \Gamma_{n+1}[\gamma,\gamma']_m$$

for all $n, m \in \mathbb{N}$ and $\gamma, \gamma' \in |\Gamma|_n$ as follows. Observe first that there is a function

$$\begin{aligned} |\Gamma|_n \times |\Gamma|_n &\to \infty\mathcal{M}_{\mathcal{U}}\mathrm{Gpd} \\ (\gamma,\gamma') &\mapsto \Gamma_{n+1}[\gamma,\gamma']. \end{aligned}$$

Because Γ is small, we can encode this function by a term

$$\Gamma_{n+1}[\_,\_] \in \mathrm{Tm}(T.\Gamma_n.\Gamma^{+}_n, \mathcal{U}^{\infty}_{k,0})$$

in $\mathcal{M}$ for some $k \in \mathbb{N}$ in the sense that it satisfies the equation

$$\mathrm{El}(\Gamma_{n+1}[\_,\_] \circ \langle \gamma,\gamma' \rangle) = \Gamma_{n+1}[\gamma,\gamma'].$$

Moreover, by the elimination rule of N-type with respect to the universe $\mathcal{U}_k$ in $\mathcal{M}$, we assemble these terms $\Gamma_{n+1}[\_,\_]$ for all $n \in \mathbb{N}$ into a single term

$$\Gamma_{\_}[\_,\_] \in \mathrm{Tm}(T.N.\Gamma.\Gamma^{+}, \mathcal{U}^{\infty}_{k,0})$$

in $\mathcal{M}$ that satisfies the equation

$$\Gamma_{\_}[\_,\_]\{\langle !, \underline{n}, \mathrm{v}\{\mathrm{p}\}, \mathrm{v} \rangle\} = \Gamma_{n+1}[\_,\_]$$

for all $n \in \mathbb{N}$. Furthermore, again by the elimination rule of N-type with respect to $\mathcal{U}_k$, it is straightforward to lift the term $\Gamma_{\_}[\_,\_]$ into another term

$$\Gamma_{\_}[\_,\_]_{\_} \in \mathrm{Tm}(T.N.\Gamma.\Gamma^{+}.N, \mathcal{U}^{\infty}_{k,0})$$

in $\mathcal{M}$ that satisfies the equation

$$\mathrm{El}(\Gamma_{\_}[\_,\_]_{\_}\{\langle !, \underline{n}, \gamma, \gamma', \underline{m} \rangle\}) = \Gamma_{n+1}[\gamma,\gamma']_m$$

for all $n, m \in \mathbb{N}$ and $\gamma, \gamma' \in |\Gamma|_n$. Finally, by applying the elimination rule of the universe $\mathcal{U}_k$ to this term $\Gamma_{\_}[\_,\_]_{\_}$, we obtain the desired type

$$\Gamma^{\star}_{\_}[\_,\_]_{\_} :\equiv \mathrm{El}(\Gamma_{\_}[\_,\_]_{\_}) \in \mathrm{Ty}(T.N.\Gamma.\Gamma^{+}.N)$$

in $\mathcal{M}$ that satisfies the equation

$$\Gamma^{\star}_{\_}[\_,\_]_{\_}\{\langle !, \underline{n}, \gamma, \gamma', \underline{m} \rangle\} = \Gamma_{n+1}[\gamma,\gamma']_m$$

for all $n, m \in \mathbb{N}$ and $\gamma, \gamma' \in |\Gamma|_n$.

Next, assume an $\mathcal{M}$-internal dependent ∞-groupoid $A : \Gamma \to \tilde{\mathcal{U}}^{\infty}$. We define a type $A^{\star}_0 \in \mathrm{Ty}(T.\Gamma_0.N)$ in $\mathcal{M}$ by

$$A^{\star}_0 :\equiv \mathrm{El}\{\mathrm{App}(\pi^{[10]}_1\{A_0 \circ \mathrm{p}\}, \mathrm{v})\}$$

and another type $A^{\star}_{0,n} \in \mathrm{Ty}(T.\Gamma_0)$ for each $n \in \mathbb{N}$ by

$$A^{\star}_{0,n} :\equiv A^{\star}_0\{\langle \mathrm{id}_{T.\Gamma_0}, \underline{n} \rangle\}.$$

Similarly, we define a type $A^\star_1 \in \mathrm{Ty}(T.\Gamma_1.N)$ in $\mathcal{M}$ by

$$A^\star_1 := \mathrm{El}\{\mathrm{App}(\pi^{[4]}_3\{A_1 \circ \mathrm{p}\}, \mathrm{v})\}.$$

Moreover, we define a type $A^\star_{m+2} \in \mathrm{Ty}(T.\Gamma_{m+2})$ in $\mathcal{M}$ for all $m \in \mathbb{N}$ by

$$A^\star_{m+2} := \mathrm{El}\{\pi^{[4]}_3\{A_{m+2}\}\}.$$

Because these types $A^\star_n$ ($n \in \mathbb{N}$) are obtained through terms in $\mathcal{M}$ by the elimination rule of a universe in $\mathcal{M}$, by applying the elimination rules of N-type of the universe to these corresponding terms, we can assemble them into a single type

$$A^\star \in \mathrm{Ty}(T.N.\mathrm{Data}_A),$$

where the type $\mathrm{Data}_A \in \mathrm{Ty}(T.N)$ is defined by the elimination rule of N-type that satisfies the equations

$$\begin{aligned}\mathrm{Data}_A\{\langle !, \underline{0}\rangle\} &= \Sigma(\Gamma_0, N)\\ \mathrm{Data}_A\{\langle !, \underline{1}\rangle\} &= \Sigma(\Gamma_1, N)\\ \mathrm{Data}_A\{\langle !, \underline{n+2}\rangle\} &= \Gamma_{n+2} \quad (n \in \mathbb{N}),\end{aligned}$$

such that for all $n \in \mathbb{N}$ the type $A^\star$ satisfies the equations

$$\begin{aligned}A^\star\{\langle !, \underline{0}, \mathrm{v}\{\mathrm{p}\}, \mathrm{v}\rangle\} &= A^\star_0\\ A^\star\{\langle !, \underline{1}, \mathrm{v}\{\mathrm{p}\}, \mathrm{v}\rangle\} &= A^\star_1\\ A^\star\{\langle !, \underline{n+2}, \mathrm{v}\rangle\} &= A^\star_{n+2} \quad (n \in \mathbb{N}).\end{aligned}$$

We are now ready to introduce:

Definition 3.26 (internal Grothendieck construction)**.** The ***Grothendieck construction*** on a small $\mathcal{M}$-internal ∞-groupoid $\Gamma \in \infty\mathcal{M}_{\mathcal{U}}\mathrm{Gpd}$ and an $\mathcal{M}$-internal dependent ∞-groupoid $A : \Gamma \to \tilde{\mathcal{U}}^\infty$ is the small $\mathcal{M}$-internal ∞-groupoid $\Gamma.A \in \infty\mathcal{M}_{\mathcal{U}}\mathrm{Gpd}$ defined as follows. We first define the type $\Gamma.A \in \mathrm{Ty}(T.N)$ by describing the types $(\Gamma.A)_n \in \mathrm{Ty}(T)$ inductively for each $n \in \mathbb{N}$, i.e., by the elimination rule of N-type with respect to a sufficiently large universe in $\mathcal{M}$, we obtain the type $\Gamma.A$ out of them.

- (0-CELLS) The type $(\Gamma.A)_0 \in \mathrm{Ty}(T)$ in $\mathcal{M}$ is defined by

$$(\Gamma.A)_0 := \mathrm{d}(T.\Gamma_0.A^\star_{0,0}).$$

- (1-CELLS) We write

$$A^\star_0(\_)_1[\_,\_] \in \mathrm{Ty}(T.\Gamma_0.A^\star_{0,0}.(A^\star_{0,0})^+)$$

  for the type in $\mathcal{M}$ that internalises the function

$$\begin{aligned}|\Gamma_0.A^\star_{0,0}.(A^\star_{0,0})^+| &\to \infty\mathcal{M}\mathrm{Gpd}_0\\ \langle \gamma, \alpha, \alpha'\rangle &\mapsto A^\star_0(\gamma)_1[\alpha, \alpha'].\end{aligned}$$

  through the evident term from the object $T.\Gamma_0.A^\star_{0,0}.(A^\star_{0,0})^+$ to a sufficiently large universe $\mathcal{U}$ in $\mathcal{M}$ (i.e., $A^\star_0(\_)_1[\_,\_]$ is obtained by applying the elimination rule of $\mathcal{U}$ to the term).

  Remark. The construction of $A^\star_0(\_)_1[\_,\_]$ is possible because $A^\star$ is already encoded by A. This is why we employ not ∞-groupoid functors $|\Gamma| \to \infty\mathcal{M}\mathrm{Gpd}$ themselves but the underlying $\mathcal{M}$-internal dependent ∞-groupoids $\Gamma \to \tilde{\mathcal{U}}^\infty$. We henceforth skip making a similar remark.

  We also write

$$\partial^{[1]} \in \mathrm{Tm}\big(T.\underline{\Gamma_0}_{(1)}.\underline{A^\star_{0,0}}_{(2)}.\underline{\Gamma^{++}_0}_{(3)}.\underline{A^\star_{0,0}}_{(4)}\{\mathrm{v}\}, \Pi(\underline{N}_{(5)}, \Gamma_1[\pi_{(1)}, \pi_{(3)}]_{\pi_{(5)}} \Rightarrow A^\star_0\{\langle \pi_{(3)}, \pi_{(5)}\rangle\}^+)\big)$$

  for the evident term that internalises the ∞-groupoid functor $|\partial|^{[1]}_{\langle\gamma,\alpha\rangle,\langle\gamma',\alpha'\rangle} : |\Gamma_1[\gamma, \gamma']| \to |A^\star_0(\gamma')|$ for each pair $\langle\gamma, \alpha\rangle, \langle\gamma', \alpha'\rangle \in |(\Gamma.A)_0| \times |(\Gamma.A)_0|$ defined by

$$\begin{aligned}|\partial|^{[1,0]}_{\langle\gamma,\alpha\rangle,\langle\gamma',\alpha'\rangle}(\sigma) &:= A^\star_{1,0}(\sigma) \circ \alpha \quad (\sigma \in |\Gamma_1(\gamma, \gamma')|_0)\\ |\partial|^{[1,m+1]}_{\langle\gamma,\alpha\rangle,\langle\gamma',\alpha'\rangle}(\tau) &:= A^\star_{m+2}(\tau) \circ \alpha \quad (m \in \mathbb{N}, \tau \in |\Gamma_1(\gamma, \gamma')|_{m+1}).\end{aligned}$$

We further define

$$\partial^{[1]}_{\langle\phi,\psi\rangle,\langle\phi',\psi'\rangle} := \partial^{[1]}\{\langle\langle\phi,\psi\rangle,\langle\phi',\psi'\rangle\rangle\}$$

for all object $\Delta \in \mathcal{M}$ and morphisms $\langle\phi,\psi\rangle,\langle\phi',\psi'\rangle : \Delta \to (\Gamma.A)_0$ in $\mathcal{M}$.
We then define $(\Gamma.A)_1 \in \mathcal{M}$ by

$$(\Gamma.A)_1 := \mathrm{d}(T.\underline{\Gamma_0}_{(1)}.\underline{A^\star_{0,0}}_{(2)}.\underline{\Gamma_0^{++}}_{(3)}.\underline{A^\star_{0,0}\{\mathrm{v}\}}_{(4)}.\Gamma_1[\pi_{(1)},\pi_{(3)}]_{\underline{0}}.A^\star_0\{\langle\pi_{(3)},\underline{1}\rangle\}[\partial^{[1,0]}_{\langle\pi_{(1)},\pi_{(2)}\rangle,\langle\pi_{(3)},\pi_{(4)}\rangle}\{\pi_{(2)}\},\pi_{(4)}]_{\underline{0}}).$$

- (Higher-cells of odd-dimension) Let $n \in \mathbb{N}$ such that $n+1$ is odd. We write

$$A^\star_{0,n+1}(\_)[\_,\_] \in \mathrm{Ty}(T.\Gamma_0.A^\star_{0,n}.(A^\star_{0,n})^+)$$

for the evident type in $\mathcal{M}$ that internalises the function

$$\begin{aligned} |T.\Gamma_0.A^\star_{0,n}.(A^\star_{0,n})^+| &\to \infty\mathcal{M}\mathrm{Gpd}_n \\ \langle\gamma,\phi,\phi'\rangle &\mapsto A^\star_0(\gamma)_{n+1}[\phi,\phi'] \end{aligned}$$

similarly to the case of $A^\star_{0,1}(\_)[\_,\_]$. We further write

$$\partial^{[n+1]} \in \mathrm{Tm}\big(T.\underline{\Gamma_n}_{(1)}.\underline{A^\star_{0,n}}_{(2)}.\underline{\Gamma_n^{++}}_{(3)}.\underline{A^\star_{0,n}\{\mathrm{v}\}}_{(4)},\Pi(\underline{N}_{(5)},\Gamma_n[\pi_{(3)},\pi_{(1)}]_{\pi_{(5)}} \Rightarrow A^\star_0(\mathrm{t}^n\circ\pi_{(1)})_n[\partial,\mathrm{t}\circ\pi_{(1)}]_{\pi_{(5)}})\big)$$

for the term that internalises the ∞-groupoid functor $|\partial|^{[n+1]}_{\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle} : |\Gamma_{n+1}[\rho,\rho']| \to |A^\star_0(\gamma')_n[\partial^{[n]}_{\langle\mu,\nu\rangle,\langle\mu',\nu'\rangle}(\rho),\nu']|$ for each pair $\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle \in |(\Gamma.A)_0| \times |(\Gamma.A)_0|$, where

$$\langle\mu,\nu\rangle := \mathrm{s}\circ\langle\rho,\phi\rangle \qquad \langle\mu',\nu'\rangle := \mathrm{t}\circ\langle\rho',\phi'\rangle$$

$$\langle\gamma,\alpha\rangle := \mathrm{s}^{n-1}\circ\langle\mu,\nu\rangle \qquad \langle\gamma',\alpha'\rangle := \mathrm{t}^{n-1}\circ\langle\mu',\nu'\rangle,$$

defined by

$$|\partial|^{[n+1]}_{\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle}(\Xi) := \mathrm{i}(\phi') *_{n-1} |\partial|^{[n]}_{\langle\mu,\nu\rangle,\langle\mu',\nu'\rangle}(\Xi) \quad (m \in \mathbb{N}, \Xi \in |\Gamma_{n+1}[\rho,\rho']|_m).$$

We also define

$$\partial^{[n+1]}_{\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle} := \partial^{[n+1]}\{\langle\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle\rangle\}$$

for any object $\Delta \in \mathcal{M}$ and morphism $\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle : \Delta \to (\Gamma.A)_n$ in $\mathcal{M}$.
We then define $(\Gamma.A)_{n+1} \in \mathcal{M}$ by

$$(\Gamma.A)_{n+1} := T.\underline{\Gamma_n}_{(1)}.\underline{A^\star_{0,n}}_{(2)}.\underline{\Gamma_n^{++}}_{(3)}.\underline{A^\star_{0,n}\{\mathrm{v}\}}_{(4)}.\underline{\Gamma_{n+1}[\pi_{(1)},\pi_{(3)}]_0}_{(5)}.A^\star_0(\mathrm{t}^n\circ\pi_{(3)})_{n+1}[\partial^{[n],0}_{\langle\pi_{(1)},\pi_{(2)}\rangle,\langle\pi_{(3)},\pi_{(4)}\rangle}\circ\pi_{(5)},\mathrm{t}\circ\pi_{(4)}]_0.$$

- (Higher-cells of even-dimension) Let $n \in \mathbb{N}$ such that $n+1$ is even. We write

$$A^\star_{0,n+1}(\_)[\_,\_] \in \mathrm{Ty}(T.\Gamma_0.A^\star_{0,n}.(A^\star_{0,n})^+)$$

for the type in $\mathcal{M}$ that internalises the function

$$\begin{aligned} |\Gamma_0.A^\star_{0,n}.(A^\star_{0,n})^+| &\to \infty\mathcal{M}\mathrm{Gpd}_n \\ \langle\gamma,\phi,\phi'\rangle &\mapsto A^\star_0(\gamma)_{n+1}[\phi,\phi'] \end{aligned}$$

just as in the previous case. We also write

$$\partial^{[n+1]} \in \mathrm{Tm}\big(T.\underline{\Gamma_n}_{(1)}.\underline{A^\star_{0,n}}_{(2)}.\underline{\Gamma_n^{++}}_{(3)}.\underline{A^\star_{0,n}}_{(4)},\Pi(N,\Gamma_{n+1}[\pi_{(1)},\pi_{(3)}]_{\mathrm{id}_N} \Rightarrow A^\star_0\{\langle\pi_{(3)},\mathrm{id}\rangle\})\big)$$

for the term that internalises the ∞-groupoid functor $|\partial|^{[1]}_{\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle} : |\Gamma_{n+1}[\rho,\rho']| \to |A^\star_0(\gamma')[\partial^{[n]}_{\langle\mu,\nu\rangle,\langle\mu',\nu'\rangle}(\rho),\nu']|$ for each pair $\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle \in |(\Gamma.A)_0| \times |(\Gamma.A)_0|$, where

$$\langle\mu,\nu\rangle := \mathrm{s}\circ\langle\rho,\phi\rangle \qquad \langle\mu',\nu'\rangle := \mathrm{t}\circ\langle\rho',\phi'\rangle$$

$$\langle\gamma,\alpha\rangle := \mathrm{s}^{n-1}\circ\langle\mu,\nu\rangle \qquad \langle\gamma',\alpha'\rangle := \mathrm{t}^{n-1}\circ\langle\mu',\nu'\rangle,$$

defined by

$$|\partial|^{[n+1]}_{\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle}(\Xi) := \mathrm{i}(\phi') *_{n-1} |\partial|^{[n]}_{\langle\mu,\nu\rangle,\langle\mu',\nu'\rangle}(\Xi) \quad (m \in \mathbb{N}, \Xi \in |\Gamma_{n+1}[\rho,\rho']|_m).$$

We further define

$$\partial^{[n+1]}_{\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle} := \partial^{[n+1]}\{\langle\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle\rangle\}$$

for all object $\Delta \in \mathcal{M}$ and morphisms $\langle\rho,\phi\rangle, \langle\rho',\phi'\rangle : \Delta \to (\Gamma.A)_n$.

We then define $(\Gamma.A)_{n+1} \in \mathcal{M}$ by

$$(\Gamma.A)_{n+1} := \underline{\Gamma_n}_{(1)}.\underline{A^\star_{0,n}}_{(2)}.\underline{\Gamma_n}_{(3)}.\underline{A^\star_{0,n}}_{(4)}.\underline{\Gamma_{n+1}[\pi_{(1)},\pi_{(3)}]_0}_{(5)}.A^\star_0(\mathrm{t}^n \circ \pi_{(3)})_{n+1}[\mathrm{s}\circ\pi_{(2)}, \partial^{[n+1],0}_{\langle\pi_{(1)},\pi_{(2)}\rangle,\langle\pi_{(3)},\pi_{(4)}\rangle} \circ \pi_{(5)}]_0.$$

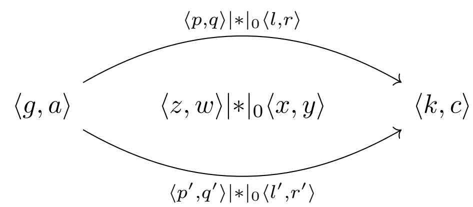
- (Sources, targets and identities) The sources and the targets in $\Gamma.A$ are the evident projections, and the identities $\mathrm{i}_n : (\Gamma.A)_n \to (\Gamma.A)_{n+1}$ $(n \in \mathbb{N})$ are defined by

$$\mathrm{i}_n := \langle \mathrm{i}, \mathrm{i}, \mathrm{i}\rangle.$$

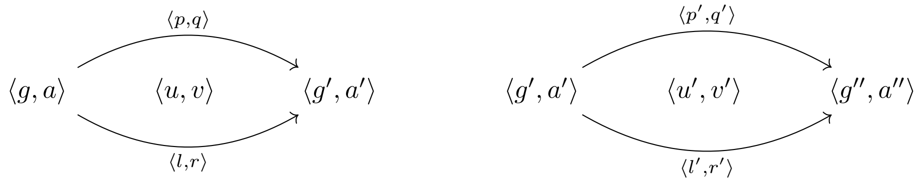
- (Horizontal composition) The horizontal composition $*_0 : (\Gamma.A)_1 \times_{(\Gamma.A)_0} (\Gamma.A)_1 \to (\Gamma.A)_1$ on 1-cells is the morphism in $\mathcal{M}$ that internalises the horizontal composition

$$\begin{aligned}&|\Gamma.A|_1 \times_{|\Gamma.A|_0} |\Gamma.A|_1 \to |\Gamma.A|_1\\ &\qquad (\langle\rho,\phi\rangle,\langle\rho',\phi'\rangle) \mapsto \langle\rho' *_0 \rho, \phi' *_0 A^\star_{1,1}(\rho')(\phi)\rangle\end{aligned}$$

in the ∞-category $|\Gamma|.|A|$ [War11, §3.1]. The horizontal composition $*_0 : (\Gamma.A)_{n+2} \times_{(\Gamma.A)_0} (\Gamma.A)_{n+2} \to (\Gamma.A)_{n+2}$ $(n \in \mathbb{N})$ on higher cells is the morphism in $\mathcal{M}$ that internalises the horizontal composition



of given $(n+2)$-cells

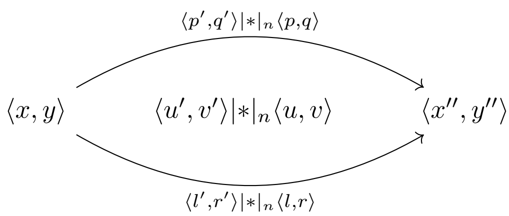


in the ∞-category $|\Gamma|.|A|$ [War11, §3.1] defined by

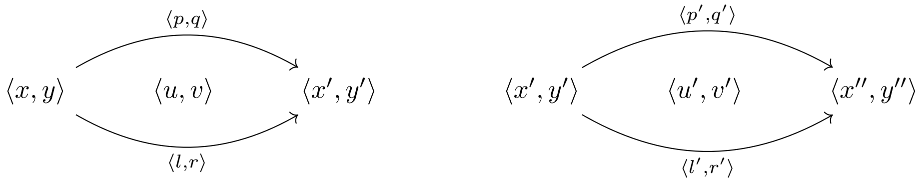
$$\begin{aligned}&|\Gamma.A|_1 \times_{|\Gamma.A|_0} |\Gamma.A|_1 \to |\Gamma.A|_1\\ &\qquad (\langle\sigma',\tau'\rangle,\langle\sigma,\tau\rangle) \mapsto \langle\sigma'|*|_0\sigma, \tau'|*|_0(A^\star_{1,n+2}(\sigma')\circ\tau)\rangle\end{aligned}$$

$$\langle u',v'\rangle|*|_0\langle u,v\rangle := \langle u'|*|_0 u, v'|*|_0(A^\star_{1,n+2}(p')\circ v)\rangle.$$

- (Higher compositions) Given positive integers $n, m \in \mathbb{N}^+$ such that $m > n$, the composition $*_n : (\Gamma.A)_m \,\&_{(\Gamma.A)_n}\, (\Gamma.A)_m \to (\Gamma.A)_m$ is the evident $\mathcal{M}$-morphism that internalises the $|*|_n$-composition



of given m-cells $\langle u, v\rangle$ and $\langle u', v'\rangle$ in $(\Gamma.A)$ bounded by $(n+1)$-cells $\langle p, q\rangle, \langle l, r\rangle : \langle x, y\rangle \to \langle x', y'\rangle$ and $\langle p', q'\rangle, \langle l', r'\rangle : \langle x', y'\rangle \to \langle x'', y''\rangle$



defined by

$$\langle u', v'\rangle|*|_n\langle u, v\rangle := \begin{cases} \langle u'|*|_n u, (v'|*|_{n-1}(|i|^{m-n} \circ \partial^{(n)}_{soy}(l)))|*|_n v\rangle & \text{if } n+1 \text{ is even;} \\ \langle u'|*|_n u, v'|*|_n((|i|^{m-n} \circ \partial^{(n)}_{toy'}(p'))|*|_{n-1}v)\rangle & \text{otherwise,} \end{cases}$$

where, if $n+1$ is even, then we have

$$\begin{aligned} |s|^{m-n}(v'|*|_{n-1}(|i|^{m-n} \circ \partial^{(n)}_{soy}(l))) &= (|s|^{m-n}(v'))|*|_{n-1}(|s|^{m-n} \circ |i|^{m-n} \circ \partial^{(n)}_{soy}(l)) \\ &= y'|*|_{n-1}\partial^{(n)}_{soy}(l) \\ &= |t|^{m-n}(v), \end{aligned}$$

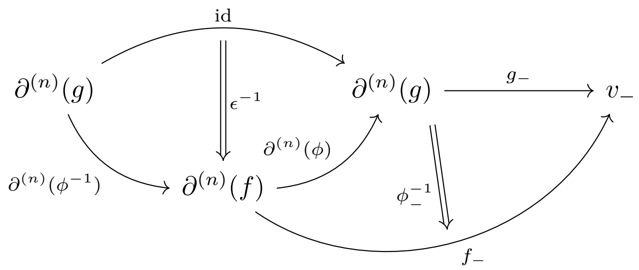
and similarly, if $n+1$ is odd, then we have

$$|s|^{m-n}(v') = |t|^{m-n}((|i|^{m-n} \circ \partial^{(n)}_{toy'}(p'))|*|_{n-1}v),$$

so that the second component of $\langle u', v'\rangle|*|_n\langle u, v\rangle$ is well-defined in either case.

At this point, let us leave it to the reader to describe the $\mathcal{M}$-morphism $*_n : (\Gamma.A)_m \&_{(\Gamma.A)_n} (\Gamma.A)_m \to (\Gamma.A)_m$ explicitly.

- (Inverses) Assume $n \in \mathbb{N}^+$. If n is even, then the inverse $\mathrm{inv}_n : (\Gamma.A)_n \to (\Gamma.A)_n$ is the evident $\mathcal{M}$-morphism that internalises the function $|(\Gamma.A)|_n \to |(\Gamma.A)|_n$ that maps a given n-cell $\langle \phi, \phi_-\rangle \in |(\Gamma.A)|_n$ to the element



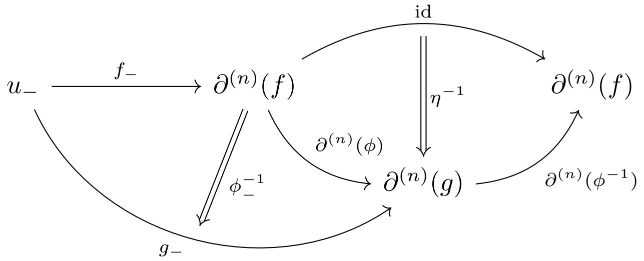
of the set $|(\Gamma.A)|_n$, where $\langle f, f_-\rangle := s(\langle \phi, \phi_-\rangle) \in |(\Gamma.A)|_{n-1}$, $\langle g, g_-\rangle := t(\langle \phi, \phi_-\rangle) \in |(\Gamma.A)|_{n-1}$, $\langle u, u_-\rangle := s(\langle f, f_-\rangle) \in |(\Gamma.A)|_{n-1}$ and $\langle v, v_-\rangle := s(\langle g, g_-\rangle) \in |(\Gamma.A)|_{n-1}$. Explicitly, $\mathrm{inv}_n : (\Gamma.A)_n \to (\Gamma.A)_n$ is the $\mathcal{M}$-pairing of

$$(\Gamma.A)_n \xrightarrow{\pi_1} \Gamma_n \xrightarrow{\mathrm{inv}} \Gamma_n$$

$$(\Gamma.A)_n \xrightarrow{*_{n-1}\circ\langle *_{n-2}\circ\langle \mathrm{inv}\circ\pi_2, i\circ\partial^{(n)}\circ\mathrm{inv}\circ\pi_1\rangle, *_{n-1}\circ\langle i\circ t\circ\pi_2, \mathrm{inv}\circ\epsilon\circ\partial^{(n)}\circ\pi_1\rangle\rangle} A^\star_{0,n}\{s^n \circ \pi_1\},$$

where the ∞-strategies assigned to the subscripts on $\partial^{(n)}$ are provided by the sources and the targets in $\mathcal{M}$.

Similarly, if n is odd, then $\mathrm{inv}_n : (\Gamma.A)_n \to (\Gamma.A)_n$ is the evident $\mathcal{M}$-morphism that internalises the function $|(\Gamma.A)|_n \to |(\Gamma.A)|_n$ that maps $\langle \phi, \phi_- \rangle$ to the element



of the set $|(\Gamma.A)|_n$. Let us leave it to the reader to describe the $\mathcal{M}$-morphism explicitly.

- (Retraction and section witnesses) The retraction witness $\eta_1 : (\Gamma.A)_1 \to (\Gamma.A)_2$ is the pairing $\langle \eta_1 \circ \pi_1, \mathrm{app}(A_{1,2} \circ \mathrm{ret}_1 \circ \pi_1, \eta_1 \circ \pi_2)\rangle$ of

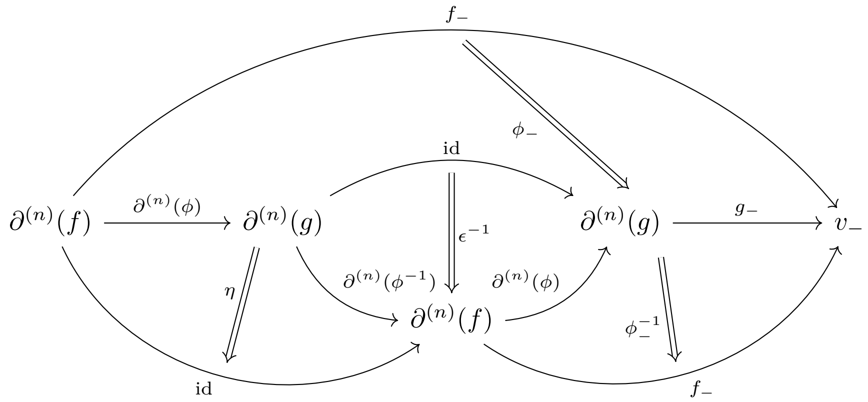
$$(\Gamma.A)_1 \xrightarrow{\pi_1} \Gamma_1 \xrightarrow{\eta_1} \Gamma_1 \qquad\qquad (\Gamma.A)_1 \xrightarrow{\mathrm{app}(A_{1,2}\circ\mathrm{ret}_1\circ\pi_1, \eta_1\circ\pi_2)} A_0^{\star}\{s\}_1.$$

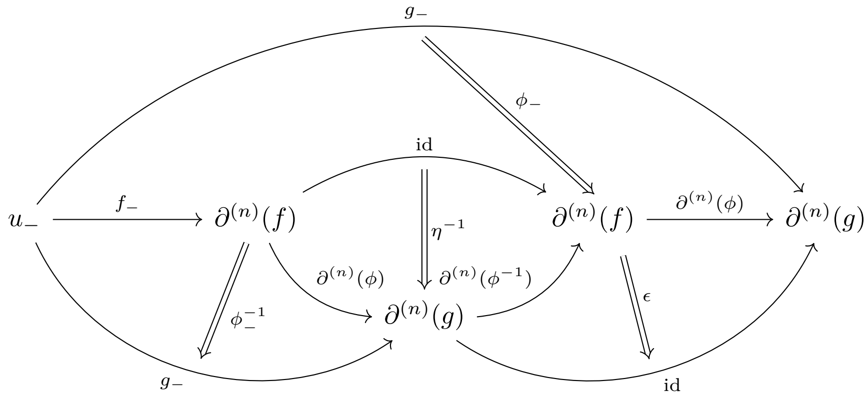
For each $n \geqslant 2$, if n is even, then the retraction witness $\eta_n : (\Gamma.A)_n \to (\Gamma.A)_{n+1}$ is the pairing $\langle \eta_n \circ \pi_1, *_{n-1} \circ \langle \mathrm{app}(\partial^{(n)}, \eta_n \circ \pi_2)\rangle\rangle$ of

$$(\Gamma.A)_1 \xrightarrow{\pi_1} \Gamma_1 \xrightarrow{\eta_1} \Gamma_1 \qquad\qquad (\Gamma.A)_1 \xrightarrow{\mathrm{app}(A_{1,1}\circ\mathrm{ret}_1\circ\pi_1, \mathrm{rec}_1\circ\pi_2)} A_0^{\star}\{s\}_1,$$

and if n is odd, then it is the pairing $\langle \mathrm{ret}_n \circ \pi_1, *_{n-1}^{n} \circ \langle \mathrm{app}(\partial^{(n)}, \mathrm{ret}_n \circ \pi_1), \mathrm{ret}_n \circ \pi_2\rangle\rangle$ of

$$(\Gamma.A)_1 \xrightarrow{\pi_1} \Gamma_1 \xrightarrow{\mathrm{ret}_1} \Gamma_1 \qquad\qquad (\Gamma.A)_1 \xrightarrow{\mathrm{app}(A_{1,1}\circ\mathrm{ret}_1\circ\pi_1, \mathrm{rec}_1\circ\pi_2)} A_0^{\star}\{s\}_1.$$





- (Delta witnesses) We leave it to the reader.

Theorem 3.27 (Well-defined Grothendieck construction)**.** *The structure* $\Gamma.A$ *forms a well-defined* $\mathcal{M}$*-internal* ∞*-groupoid.*

4 Internal ∞-groupoid interpretation of homotopy type theory

4.1 Pi

Definition 4.1 (Pi)**.** The *pi* from a small $\mathcal{M}$-internal ∞-groupoid $\Gamma \in \infty\mathcal{M}_{\mathcal{U}}\mathrm{Gpd}$ to an $\mathcal{M}$-internal dependent ∞-groupoid $A : \Gamma \to \tilde{\mathcal{U}^\infty}$ is a small $\mathcal{M}$-internal ∞-groupoid $\Pi(\Gamma, A) \in \infty\mathcal{M}_{\mathcal{U}}\mathrm{Gpd}$ defined as follows:

- (0-cells) $\Pi(\Gamma, A)_0 \in \mathcal{M}$ is defined by

$$\Pi(\Gamma, A)_0 := \{\, f \in \mathrm{Tm}(N, \Pi(\Gamma, A^\star)) \mid \overline{f} := \langle \mathrm{pId}_\Gamma, (f_n := f \circ \underline{n})_{n \in \mathbb{N}} \rangle : \Gamma \to \Sigma(\Gamma, A) \,\},$$

  where $\Pi(\Gamma, A^\star)$ is the $\mathcal{M}$-type on the natural number game N defined by

$$\Pi(\Gamma, A^\star)(\underline{n}) := \Pi(\Gamma_n, A^\star_n) \quad (n \in \mathbb{N}).$$

- (1-cells) $\Pi(\Gamma, A)_1 \in \mathcal{M}$ is defined by

$$\Pi(\Gamma, A)_1 := \int_{f, g \in \Pi(\Gamma, A)_0} \{\, p \in \mathrm{Tm}(\Gamma_0, A^\star_{0,1}(f_0, g_0)_0) \mid \overline{p} := \langle i, p \rangle : \overline{f} \to \overline{g} \,\},$$

  where $A^\star_{0,1}(f_0, g_0)_0$ is the $\mathcal{M}$-type on Γ_0 defined by

$$A^\star_{0,1}(f_0, g_0)_0(\gamma) := A^\star_0(\gamma)_1(f_0 \circ \gamma, g_0 \circ \gamma)_0 \quad (\gamma \in |\Gamma|_0).$$

- (Higher-cells) $\Pi(\Gamma, A)_n \in \mathcal{M}$ $(n \geqslant 2)$ is defined by

$$\Pi(\Gamma, A)_n := \int_{p, q \in \Pi(\Gamma, A)_{n-1}} \{\, u \in \mathrm{Tm}(\Gamma_0, A^\star_{0,n}(p, q)_0) \mid \overline{u} := \langle i^n, u \rangle : \overline{p} \to \overline{q} \,\},$$

  where $A^\star_{0,n}(p, q)_0$ is the $\mathcal{M}$-type on Γ_0 defined by

$$A^\star_{0,n}(p, q)_0(\gamma) := A^\star_0(\gamma)_n(p \circ \gamma, q \circ \gamma)_0 \quad (\gamma \in |\Gamma|_0).$$

- (Compositions) Let $n, p \in \mathbb{N}$ such that $p < n$. The composition $*_p : \Pi(\Gamma, A)_n \,\&_{\Pi(\Gamma,A)_p} \Pi(\Gamma, A)_n \to \Pi(\Gamma, A)_n$ is the currying of the $\mathcal{M}$-morphism

$$\begin{array}{ccc} (\Pi(\Gamma, A)_n \,\&_{\Pi(\Gamma,A)_p} \Pi(\Gamma, A)_n) \,\&\, \Gamma_0 & \xrightarrow{\Delta} & (\Sigma(\Gamma, A)_n^{\Gamma_0} \,\&\, \Gamma_0) \,\&_{\Sigma(\Gamma,A)_p} (\Sigma(\Gamma, A)_n^{\Gamma_0} \,\&\, \Gamma_0) \\ & & \big\downarrow \mathrm{ev} \,\&_{\Sigma(\Gamma,A)_p} \mathrm{ev} \\ \int A^\star_{0,n} & \xleftarrow[\pi_2 \circ *_p]{} & \Sigma(\Gamma, A)_n \,\&_{\Sigma(\Gamma,A)_p} \Sigma(\Gamma, A)_n \end{array}$$

where the $\mathcal{M}$-morphism Δ is the diagonal on Γ_0 up to the evident $\mathcal{M}$-isomorphism.

- (Identities) The identity $i_n : \Pi(\Gamma, A)_n \to \Pi(\Gamma, A)_{n+1}$ ($n \in \mathbb{N}$) is the $\mathcal{M}$-morphism that computes as the dereliction $\mathrm{der}_{\int A^\star_{n+1}}$ up to 'tags' on the codomain.

- (Inverses) The inverse $\mathrm{inv}_{n+1} : \Pi(\Gamma, A)_{n+1} \to \Pi(\Gamma, A)_{n+1}$ ($n \in \mathbb{N}$) is the $\mathcal{M}$-morphism

$$\Pi(\Gamma, A)_{n+1} \xrightarrow{\langle i^{n+1}, \_ \rangle} (\Gamma.A)^{\Gamma_0}_{n+1} \xrightarrow{\mathrm{inv}^{\mathrm{pId}}_{n+1}} (\Gamma.A)^{\Gamma_0}_{n+1} \xrightarrow{\pi_2\{\_\}} \Pi(\Gamma, A)_{n+1}.$$

- ((Co)units) The unit $\eta_{n+1} : \Pi(\Gamma, A)_{n+1} \to \Pi(\Gamma, A)_{n+2}$ ($n \in \mathbb{N}$) is the $\mathcal{M}$-morphism

$$\Pi(\Gamma, A)_{n+1} \xrightarrow{\langle i^{n+1}, \_ \rangle} (\Gamma.A)^{\Gamma_0}_{n+1} \xrightarrow{\eta^{\mathrm{pId}}_{n+1}} (\Gamma.A)^{\Gamma_0}_{n+2} \xrightarrow{\pi_2\{\_\}} \Pi(\Gamma, A)_{n+2},$$

and the counit $\epsilon_{n+1} : \Pi(\Gamma, A)_{n+1} \to \Pi(\Gamma, A)_{n+2}$ is given analogously.

- (Triangle transformation) The triangle transformation $\delta_{n+1} : \Pi(\Gamma, A)_{n+1} \to \Pi(\Gamma, A)_{n+3}$ ($n \in \mathbb{N}$) is the $\mathcal{M}$-morphism

$$\Pi(\Gamma, A)_{n+1} \xrightarrow{\langle i^{n+1}, \_ \rangle} (\Gamma.A)^{\Gamma_0}_{n+1} \xrightarrow{\delta^{\mathrm{pId}}_{n+1}} (\Gamma.A)^{\Gamma_0}_{n+3} \xrightarrow{\pi_2\{\_\}} \Pi(\Gamma, A)_{n+3}.$$

Theorem 4.2 (Well-defined pi on ∞-games)**.** *The pi $\Pi(\Gamma, A)$ of any ∞-game $\Gamma \in \infty\mathcal{M}\mathrm{Gpd}$ and dependent ∞-game A on Γ is a well-defined ∞-game.*

4.2 Identities

We basically follow Warren [War11]. The details will appear shortly.

4.3 Constant types

The details will appear shortly.

4.4 An interpretation of homotopy type theory

In the same vein, we can internalise Warren's interpretation, showing:

Theorem 4.3 (Internal ∞-grouoid interpretation of homotopy type theory)**.** *The category $\infty\mathcal{M}\mathrm{Gpd}$ gives rise to a CwF that supports One-, Zero-, N-, Pi-, Sigma- and Id-types as well as the cumulative hierarchy of universes.*

5 Game semantics of homotopy type theory

We obtain game semantics of HoTT simply by specialising in the case $\mathcal{M} = \mathcal{G}$ [Yam19].

6 Independence of Markov's principle

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