Dependent Types and Finite Limits in Games

Norihiro Yamada

yamad041@umn.edu
University of Minnesota

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Goal and plan of the talk

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1. Introduction to game semantics and why it matters
2. Challenges in game semantics of dependent types
3. Main ideas in my solution
4. Ongoing and future research

Game semantics: why does it matter?

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- As an inspiration for new categories and types;
- For the meta-theoretic study of type theory (e.g., independence of Markov's principle);
- As pure mathematics of logic and computation in its own right: *algorithms*, *normalisation*, *higher-order computability*, etc.

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If you are working on these topics, we should collaborate!

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- Syntactic and inductive: It models types and terms by *lists* of games and strategies, respectively.

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Corollary (game semantics of homotopy type theory)

There is game semantics of homotopy type theory.

Games and strategies (1/2)

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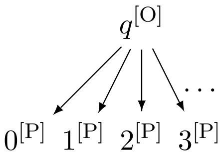

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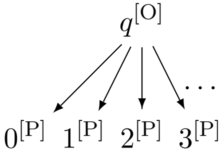

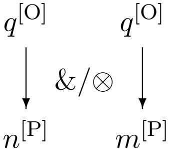

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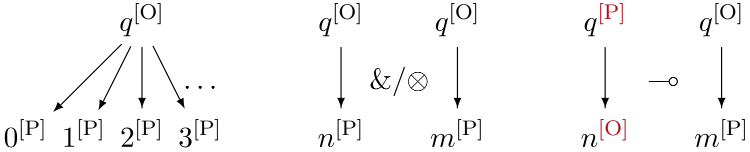

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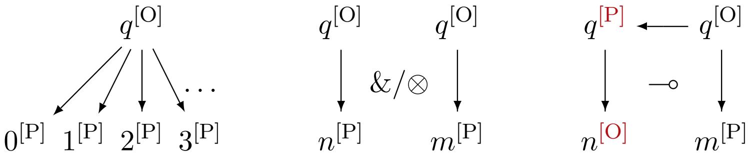

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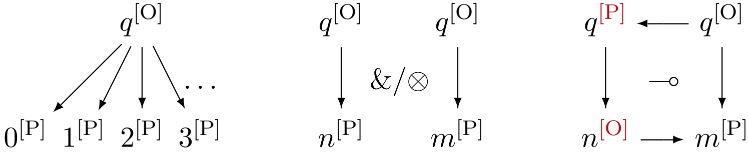

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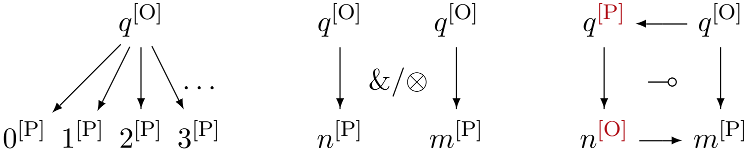

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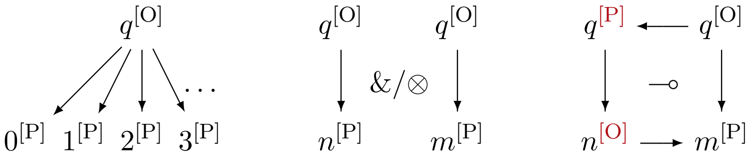

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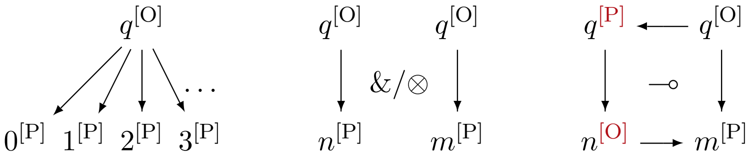

Definition (product, tensor and linear implication on games)

Product: $G \,\&\, H := G \uplus H$;
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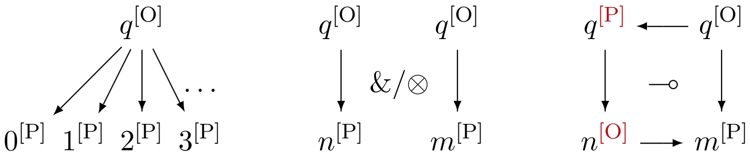

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Games and strategies (2/2)

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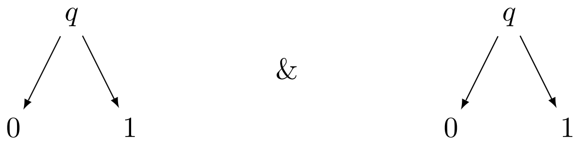

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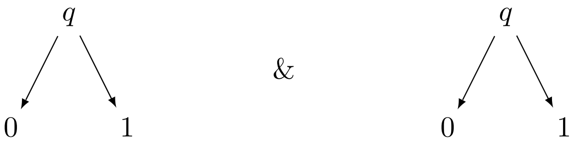

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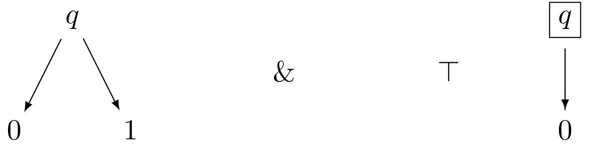

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$$\frac{G}{q_G}$$

What is your strategy?

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G	
q_G	What is your strategy?
σ	It is $\sigma : G$, which satisfies the axiom (1)!

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m_1	

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G	
q_G	What is your strategy?
σ	It is $\sigma : G$, which satisfies the axiom (1)!
m_1	
m_2	

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$$\begin{array}{c} G \\ \hline q_G \\ \sigma \\ m_1 \\ m_2 \\ \vdots \end{array}$$

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q_G	What is your strategy?
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m_2	
$\vdots$	($m_1 m_2 \dots$ is played by σ in $f_G(\sigma) \subseteq G$)

My solution: strategy filtering and game changing (2/3)

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For instance, we can interpret the Sigma-type $\Sigma_{\mathsf{x}:\mathsf{N}}\mathsf{N}_{\mathsf{b}}(\mathsf{x})$ by the pair $\Sigma(N, N_b) = (N \mathbin{\&} N, f_{\Sigma(N,N_b)})$, where $f_{\Sigma(N,N_b)} : \langle \underline{k}, \underline{n} \rangle \mapsto N \mathbin{\&} N_b(\underline{k})$.

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$$
\begin{array}{ccc}
\Sigma(N & , & N_b) \\
\hline
& q_{\Sigma(N,N_b)} & \\
& \langle \underline{1}, \underline{0} \rangle & \\
q & & \\
1 & &
\end{array}
\qquad
\begin{array}{ccc}
\Sigma(N & , & N_b) \\
\hline
& q_{\Sigma(N,N_b)} & \\
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& & 0
\end{array}
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In contrast, the pairing $\langle \underline{0}, \underline{1} \rangle$ is prohibited by $f_{\Sigma(N,N_b)}(\langle \underline{0}, \underline{1} \rangle) = N \,\&\, N_b(\underline{0})$ since $\underline{1}$ violates the rules of $N_b(\underline{0}) = \mathrm{Pref}(\{q0\})$.

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Intuition: One may think of the map f_G as giving an *additional specification* of the rules of the game G that filters strategies $\sigma : G$.

My solution: strategy filtering and game changing (3/3)

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For another example, we can interpret the Sigma-type $\Sigma_{\mathsf{x}:\mathsf{N}}\mathsf{List}_{\mathsf{N}}(\mathsf{x})$ by the pair $\Sigma(N, \mathrm{List}_N) = (N \mathbin{\&} \bigcup_{k \in \mathbb{N}} \mathrm{List}_N(\underline{k}), f_{\Sigma(N,\mathrm{List}_N)})$, where $f_{\Sigma(N,\mathrm{List}_N)} : \langle \underline{k}, \underline{n_1} \otimes \underline{n_2} \otimes \dots \rangle \mapsto N \mathbin{\&} \mathrm{List}_N(\underline{k})$.

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q

2

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q

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q

2

$$\frac{\Sigma(N \qquad , \qquad \mathrm{List}_N)}{\begin{matrix} q_{\Sigma(N,\mathrm{List}_N)} \\ \langle \underline{2}, \underline{0} \otimes \underline{1} \rangle \end{matrix}}$$

q

My solution: strategy filtering and game changing (3/3)

For another example, we can interpret the Sigma-type $\Sigma_{\mathsf{x:N}}\mathsf{List_N}(\mathsf{x})$ by the pair $\Sigma(N, \mathrm{List}_N) = (N \,\&\, \bigcup_{k \in \mathbb{N}} \mathrm{List}_N(\underline{k}), f_{\Sigma(N,\mathrm{List}_N)})$, where $f_{\Sigma(N,\mathrm{List}_N)} : \langle \underline{k}, \underline{n_1} \otimes \underline{n_2} \otimes \dots \rangle \mapsto N \,\&\, \mathrm{List}_N(\underline{k})$.

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\qquad
\frac{\Sigma(N \qquad , \qquad \mathrm{List}_N)}{\begin{array}{r} q_{\Sigma(N,\mathrm{List}_N)} \qquad\qquad \\ \langle \underline{2}, \underline{0} \otimes \underline{1} \rangle \qquad\qquad \\ q \\ 1 \end{array}}$$

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$\Sigma(N \quad , \quad \mathrm{List}_N)$

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	$q_{\Sigma(N,\mathrm{List}_N)}$	
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q		
2		

$\Sigma(N$	,	$\mathrm{List}_N)$	
	$q_{\Sigma(N,\mathrm{List}_N)}$		
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		q	
		0	

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The declaration of the strategy $\langle \underline{2}, \underline{0} \otimes \underline{1} \rangle : N \,\&\, \bigcup_{k \in \mathbb{N}} \mathrm{List}_N(\underline{k})$ fixes the underlying game $N \,\&\, (N \otimes N) \subseteq N \,\&\, \bigcup_{k \in \mathbb{N}} \mathrm{List}_N(\underline{k})$, so that $\langle \underline{2}, \underline{0} \otimes \underline{1} \rangle$ is total without redundant computation.

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Intuition: One may think of the map f_G as *giving an additional power for Player to change the rules* of the game G.

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$$
\begin{array}{ccc}
N \multimap N & \multimap & N \multimap N \\
\hline
 & q_{(N \multimap N) \multimap (N \multimap N)} & \\
 & \mathrm{succ} \circ (\_) & \\
 & & q_{N \multimap N} \\
q_{N \multimap N} & & \\
\sigma & & \\
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the intensionality of game semantics would *collapse*.

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σ

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$\vdots$

the intensionality of game semantics would *collapse*.
However, without a declaration of $\sigma : N \multimap N$ on the domain by Opponent, *what is the underlying game $f_{N \multimap N}(\sigma)$ on the domain?*

Pi-types in predicate games (1/3)

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$\Sigma(N,$	$\mathrm{List}_N)$	$\multimap$	N

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$\Sigma(N,$	$\mathrm{List}_N)$	$\multimap$	N
		$q_{\Delta\multimap\Gamma}$	
		π_1	
			q
q			
k			
			k

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Pi-types in predicate games (1/3)

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$$\frac{\Pi_\ell(N, \qquad\qquad\qquad \mathrm{List}_N)}{q_{\Pi_\ell}}$$

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q_{Π_ℓ}

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q (in List_N)

q (in N)

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		q
q		
1		

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q	
1	
	0

$\Pi_\ell(\Sigma(N,$	$N_b)$	,	$N_b\{\pi_1\})$

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1		
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1		
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The intensionality of game semantics is preserved.

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We finally define the ***pi*** Π by $\Pi(\Gamma, B) := \Pi_\ell(!\Gamma, B)$, where B is over $!\Gamma$.

Pi-types in predicate games (3/3)

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Definition (Pi-types in predicate games)

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$$\cup \{\, \boldsymbol{t}lr \in |\Pi_\ell(\Gamma, A)|^{\mathrm{Even}} \mid \boldsymbol{t}l \in \Pi_\ell(\Gamma, A)(\phi), \forall \gamma : \Gamma.\, \boldsymbol{t}l \in A(\gamma)(\phi \circ \gamma)^{\overline{\gamma}\Gamma} \Rightarrow \boldsymbol{t}lr \in A(\gamma)(\phi \circ \gamma)^{\overline{\gamma}\Gamma} \,\}$$

for all $\phi : |\Pi_\ell(\Gamma, A)|$.

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Theorem (finite limits in predicate games)

The categories of p-games and strict strategies are finitely complete.

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Proof (sketch).

The equaliser of given morphisms $\phi_1, \phi_2 : \Gamma \rightrightarrows \Delta$ is the p-game Θ defined by $|\Theta| := |\Gamma|$ and $f_\Theta(\theta) := \begin{cases} f_\Gamma(\theta) & \text{if } \phi_1 \bullet \theta = \phi_2 \bullet \theta \\ T & \text{otherwise} \end{cases}$ for all $\theta : |\Theta|$, together with the identity $\mathrm{id}_{|\Theta|} : \Theta \hookrightarrow \Gamma$. □

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Then observe that there is no *winning* morphism $(N \multimap N) \to \Omega$ that characterises the subobject $(N \multimap N)^+$ since *there is no finite interaction with a given $\phi : N \multimap N$ that decides if ϕ never outputs $\underline{0}$*.

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