Categorical algebras for linearity and dependency

Norihiro Yamada

yamad041@umn.edu
School of Mathematics
University of Minnesota

June 5, 2023

Abstract

On the one hand, *linear functions* arise in various fields of mathematics and beyond, e.g., linear algebra, functional analysis, representation theory, logic and quantum physics. Also, linear logic defines *linear proofs* and shows that linearity does not restrict but *refines* logical constructions by decomposing them into more primitive ones. On the other hand, predicates, or more generally *dependent types*, are indispensable part of foundations of mathematics since without them one cannot even talk about properties of individual objects. It is then a natural aim to blend linearity and dependency in terms of the general framework of category theory because, in addition to clarifying how these two fundamental concepts interact, the combination will analyse and polish dependency through the lens of linearity and provide a mathematical universe to reason about linear functions and proofs. However, this blending is notoriously difficult, and a solution to this problem has not been established for a long time. The present work addresses this well-known problem via *modules* (for linearity) in the setting of *indexed categories* (for dependent types). Specifically, we introduce *indexed module categories* with *monoidal comprehension* as a categorical blend of linearity and dependency, and for their reasonability prove that they are sound and complete for a linear refinement of Martin-Löf type theory, a prominent foundation of mathematics. We also show that vector spaces form an example of this blending. This result reveals a module structure underlying dependent types and shows that linearity in algebra coincides with that in logic.

Contents

1 Introduction

1.1 Linearity and dependency

On the one hand, *linear functions* play key roles in many branches of mathematics and beyond, e.g., linear algebra, functional analysis, representation theory, logic and quantum physics. Their standard formalism is vector spaces, or more generally, *modules*. Also, Girard's linear logic [Gir87] introduces *linear proofs* and refines existing logics by decomposing their logical constructions into more primitive ones. Mathematically, the refinement (or translation) of intuitionistic logic into intuitionistic linear logic is characterised by an adjunction [BBDPH93].

On the other hand, *predicates* constitute indispensable part of foundations of mathematics by expressing properties of individual objects. In the framework of *type theories* [Chu40, SU06], a class of formal systems as well as functional programming languages, one may consider a natural generalisation of predicates known as *dependent types* [Hof97]. In terms of the standard, model-theoretic interpretation, a predicate is a relation or a boolean-valued function, while a dependent type is a set-indexed family of sets or an arbitrary function.

Why does one care such a generalisation? First, it is a mathematically natural one because it corresponds to the generalisation of boolean-valued maps to general maps. Second, recall that type theories serve as computational foundations of mathematics [ML75, ML84b, CH88] with applications to programming [CAB$^+$86, Sch09] via the *proofs-as-programs* paradigm [ML84a]. By admitting *nontrivial* proofs beyond boolean values, dependent types play central roles in this computational paradigm. Lastly, this type-theoretic foundation has been extended significantly by Voevodsky et al. under the name of *homotopy type theory* and *univalent foundations* [Uni13]. This extension has created a very active field of research, forming a beautiful interface between type theory, homotopy theory and higher category theory. This new connection is based on

the *higher-dimensional* structure underlying dependent types [HS98, vdBG11, Lum09, War11, AW09, KL21], but this structure becomes trivial if one focuses on predicates.

1.2 Our goal: a categorical blend of linearity and dependency

There are compelling reasons for aiming to combine linearity and dependency. First of all, it is an intriguing mathematical problem in its own right to blend these two fundamental concepts. As explained below, this problem poses a technical challenge. Second, the blend will advance an analysis on dependent types through the lens of linearity by decomposing operations on dependent types into more primitive ones as in linear logic (n.b., linear logic has refined *propositional* logics and *simple* type theories, while our aim is to refine more general *predicate* logics and *depenent* type theories). Also, such an analysis can be a step towards an extension of the type-theoretic foundations of mathematics *constructively* to classical reasoning as linear logic uncovers a constructive interpretation of classical propositional logic [Gir87]. Such an extension will be an innovation as it is still poorly understood how to combine dependent types constructively with classical logic, e.g., see [Her05]; this problem is one of the main bottlenecks in restoring classical mathematics constructively. Last but not least, a blend of linearity and dependency will lead to a powerful foundation of mathematics that can reason about linear functions and proofs.

While logic and type theory motivate this work to a large extent, linearity and dependency are both quite general concepts not specific to logic, type theory or even set theory. Our standpoint is that *category theory* [ML13] is general enough to formulate these ubiquitous concepts. Hence, we aim to blend linearity and dependency in terms of category theory, and show its reasonability concretely in relation to a type-theoretic blend of linear logic and dependent types. We shall also prove that well-known 'linear spaces' in mathematics such as vector spaces, Banach spaces and Hilbert spaces constitute instances of our categorical framework.

1.3 Past attempts and obstacles

However, it is notoriously difficult to combine linear logic and dependent types, both categorically and type-theoretically, and it has been a long-standing problem in the fields. Indeed, this problem has been unsolved for nearly thirty years since the initial attempt [CP96] though some progresses have been made by various researchers [KPB15, GL12, SHU13, PS12, Vák15, McB16, Atk18].

Why is the combination hard to accomplish? We shall answer this question in terms of both type theory and category theory. Let us first explain it in terms of type theory. Recall that the components of a type theory are a *context* Γ, a *(dependent) type* A over a context Γ, written $\Gamma \vdash \mathsf{A}\ \mathsf{type}$, and a *term* a from a context Γ to a type A over Γ, written $\Gamma \vdash \mathsf{a} : \mathsf{A}$. From the logical point of view, a context Γ represents an *assumption*, a type A over Γ a *(generalised) predicate* over Γ, and a term a from Γ to A a *proof* of A under Γ. The model- or set-theoretic semantics $[\![\_]\!]$ interprets a context Γ by a set $[\![\Gamma]\!]$, a type $\Gamma \vdash \mathsf{A}\ \mathsf{type}$ by a set-indexed family $[\![\mathsf{A}]\!] = \{[\![\mathsf{A}]\!]_\gamma\}_{\gamma \in [\![\Gamma]\!]}$ of sets $[\![\mathsf{A}]\!]_\gamma$, and a term $\Gamma \vdash \mathsf{a} : \mathsf{A}$ by a map $[\![\mathsf{a}]\!] : [\![\Gamma]\!] \to \bigcup_{\gamma \in [\![\Gamma]\!]} [\![\mathsf{A}]\!]_\gamma$ such that $[\![\mathsf{a}]\!](\gamma) \in [\![\mathsf{A}]\!]_\gamma$ for all $\gamma \in [\![\Gamma]\!]$. A type is said to be *simple* if it does not contain a variable. Simple types generalise constant predicates or *propositions*, and the set-theoretic semantics interprets them as singleton families of sets. Every type S over the *empty context* $(\_)$ is simple since $(\_)$ does not contain any variable; we abbreviate $(\_) \vdash \mathsf{S}\ \mathsf{type}$ as S. For instance, the set-theoretic semantics interprets a simple type N of natural numbers and a type $\mathsf{N} \vdash \mathsf{List}_{\mathsf{N}}\ \mathsf{type}$ of finite lists of natural numbers by $[\![\mathsf{N}]\!] := \mathbb{N}$ and $[\![\mathsf{List}_{\mathsf{N}}]\!] := \{\mathbb{N}^k\}_{k \in \mathbb{N}}$, where X^k denotes the k-ary cartesian product of a set X.

Now, recall that linearity of a proof in the sense of linear logic is reflected in a type theory by the property of a term that it contains precisely one copy of each variable in the context. For instance, the term $\mathsf{x} : \mathsf{N}, \mathsf{y} : \mathsf{N} \vdash \mathsf{x} + \mathsf{y} : \mathsf{N}$ is linear, but the one $\mathsf{x} : \mathsf{N} \vdash \mathsf{x} + \mathsf{x} : \mathsf{N}$ is not. Then,

a main problem in blending linearity in this sense and dependent types is that a variable may occur not only in a term *but also in a dependent type*. Note that this problem is irrelevant to simple types. For example, consider the following term (which is taken from [Atk18, §1]):

$$\mathsf{x} : \mathsf{N}, \mathsf{y} : \mathrm{List}_{\mathsf{N}}(\mathsf{x}) \vdash \mathsf{y} : \mathrm{List}_{\mathsf{N}}(\mathsf{x}). \tag{1}$$

This term y is not linear because the variable x in the context does not occur in the term y, and because x occurs twice in $\mathsf{List}_{\mathsf{N}}(\mathsf{x})$. Nevertheless, the computation of this term y is essentially the identity map, and the variable x in the context is not a computational input for y. Hence, one should be able to 'linearise' this term y by removing x in the context, but it is impossible since x plays an indispensable role in the term as a *parameter* for the dependent type $\mathsf{List}_{\mathsf{N}}(\mathsf{x})$. In this way, variables in dependent types make it hopeless to realise a linear term of a dependent type.

Next, to depict the same problem from a categorical angle, recall that a standard categorical characterisation of dependent types is Jacobs' *comprehension categories* [Jac93] or any other equivalent categorical structures [Hof97, §3.2]. The set-theoretic semantics forms their instance. A comprehension category interprets the context $\mathsf{x} : \mathsf{N}, \mathsf{y} : \mathsf{List}_{\mathsf{N}}(\mathsf{x})$ as the (generalised) cartesian product of N and List, the term $\mathsf{x} : \mathsf{N}, \mathsf{y} : \mathsf{List}_{\mathsf{N}}(\mathsf{x}) \vdash \mathsf{x} : \mathsf{N}$ as the first projection of the cartesian product, and the above term (1) as the second projection. Moreover, a comprehension category interprets dependent sum and product types, or generalised existential and universal quantifiers, respectively, by adjoints to the indexing functors induced by the first projections. Therefore, the base category of a comprehension category has finite (generalised) products. However, projections *discard* inputs, and pairings *contract* them. Thus, the basic structure of a comprehension category is already incompatible with linearity in logic. Besides, as non-cartesian categories include those whose morphisms are a class of linear functions, e.g., the category of Hilbert spaces, the cartesian structure makes it hopeless for comprehension categories to embrace linearity in algebras too.

In the following, we list some attempts to address this problem in the literature of mathematics and computer science, and explain why they do not solve the problem completely.

1.3.1 Dual contexts

A major type-theoretic method adopted by Cervesato and Pfenning [CP96], Krishnaswami et al. [KPB15] and Vákàr [Vák15] for circumventing the conflict between linearity and dependency is to split a context into two regions, *cartesian* and *linear* ones, by a semicolon, and allow a type to contain only variables in the cartesian region. This *dual context* method is originally invented by Barber and Plotkin [BP96] on a simply-typed calculus for intuitionistic propositional linear logic. For instance, this approach transforms the term (1) into the form

$$\mathsf{x} : \mathsf{N}; \mathsf{y} : \mathsf{List}_{\mathsf{N}}(\mathsf{x}) \vdash \mathsf{y} : \mathsf{List}_{\mathsf{N}}(\mathsf{x}), \tag{2}$$

in which the left- (respectively, right-) hand side of the semicolon is the cartesian (respectively, linear) region. One can then apply the concept of linearity of a term to this dual context method by focusing on variables in the linear region. In this sense, the term (2) is linear.

In contrast, variables in the cartesian region are discardable and contractible because the reasoning about terms by a dependent type must be cartesian. For instance, the dependent type $\mathsf{f}, \mathsf{g} : \mathsf{N} \multimap \mathsf{N}, \mathsf{x} : \mathsf{N} \vdash \mathsf{f}(\mathsf{f}(\mathsf{x})) < \mathsf{g}(\mathsf{x})\ \mathsf{type}$, where $\multimap$ represents linear maps, and $<$ the standard order between natural numbers, should be a basic vocabulary in a linear dependent type theory, and it *duplicates* the variables f and x, respectively. I.e., dependent types must be cartesian for a type theory to be expressive even if terms are required to be non-cartesian.

However, this approach does not realise a *true interaction* between linearity and dependency since the two regions are completely separated; i.e., it is only a *disjoint union* of linear logic and

a dependent type theory. For instance, this method cannot define a linear variant of dependent sum or product types because a type cannot vary over variables in the linear region. For the same reason, it cannot reason properly about dependent tensor or linear maps either. As a result, the dual context system is hopeless with our aim to refine dependency by linearity or to extend the type-theoretic foundations of mathematics by reasoning about linear maps and proofs.

Unsurprisingly, the categorical counterpart of the dual context linear dependent type theory given by Vákár [Vák15] is essentially the disjoint union of the existing categorical semantics of intuitionistic linear logic [BBDPH93] and dependent type theories [Jac93]. Needless to say, this approach therefore does not depict how linearity interact with dependency in a genuine sense.

1.3.2 Quantitative type theories

Dual contexts were the best possible approaches to linear dependent type theories until McBride [McB16] made a breakthrough by adapting the *usage annotation* by semirings [BGMZ14, POM14, GS14] to a dependent type theory. The annotation is to assign an element n of a semiring to each variable in a context, which means that the variable is to be consumed precisely n times. The use of a semiring is suited here as type-theoretic constructions require addition and multiplication on annotated elements. McBride's innovative idea is then to regard variables used by a dependent type as *consumed 0-times*. For instance, his method transforms the term (1) into the one

$$\mathsf{x}^0 : \mathsf{N}, \mathsf{y}^1 : \mathsf{List}_{\mathsf{N}}(\mathsf{x}^0) \vdash \mathsf{y}^1 : \mathsf{List}_{\mathsf{N}}(\mathsf{x}^0), \tag{3}$$

where the superscripts on variables are elements of a fixed semiring. The total number of consumed variables matches that of produced variables: $0 = 0 + 0$ for x, and $1 = 1$ for y. Thus, the modified term (3) is *linear* in the sense of linear logic (i.e., faithful to the resource requirement of the context). In this way, McBride realised a linear dependent type theory in which a type may depend on variables classified as those in the linear region in the dual context system (§1.3.1).

However, it has turned out that McBride's linear dependent type theory has a fundamental flaw due to its linear pi-types: not closed under substitution. This problem corresponds in category theory to the ill-definedness of composition of morphisms. This problem is fixed by Atkey [Atk18] by restricting the annotation on a term to 0 or 1. Unfortunately, this restriction prohibits terms from possessing quantitative information. This restriction also makes it hopeless to regard terms as linear maps because it bans addition or scalar multiplication on terms. Consequently, the linearity of a term in the sense of Atkey diverges from the standard one in mathematics.

Finally, no categorical semantics of McBride's or Atkey's type theory has been established (though Atkey built equational semantics [Atk18, §3]). Thus, the problem of combining linearity and dependency in the general framework of category theory is still open.

1.4 Main results and our contributions

The present work establishes a true blend of linearity and dependency in terms of type theory and category theory as follows. We first define the syntax of a linear variant of *Martin-Löf type theory (MLTT)* [ML82, ML84b, ML98] or one of the best-known dependent type theories, called *linear MLTT (LMLTT)*. LMLTT is equipped with the dependent-type generalisations of logical constructions in intuitionistic linear logic and identity types. We then prove some basic properties of LMLTT; in particular, we show that LMLTT is closed under substitution, overcoming the deficiency of McBride, and further that it gives rise to the categorical structure sketched below. LMLTT is also free from Atkey's undesirable restriction on the annotation of terms. One of the key ideas that enables this innovation is to apply the structure of *modules* (in the sense of linear algebra). Moreover, by a generalisation of Girard's translation from intuitionistic linear logic to

intuitionistic logic [Gir87], LMLTT recovers MLTT; i.e., LMLTT is a linear refinement of MLTT, where constructions in the former refines those in the latter.

Next, we introduce a categorical reformulation of modules over a semiring, generalise them to indexed ones and equip them with a linear refinement of comprehension [Jac93]. We further define categorical constructions in this framework that correspond to type constructions in LMLTT. The resulting structure is called *indexed module categories* with *monoidal comprehension*, or *module comprehension categories* for short. As a main result, we show that module comprehension categories yield sound and complete semantics of LMLTT; i.e., they achieve a categorical blend of linearity and dependency that matches the type-theoretic one. The soundness and the completeness also imply that our formulation of linearity in the sense of linear algebra (or modules) coincides with that in linear logic (or type theory).

In addition, we lift the linear/non-linear adjunction [BBDPH93], the categorical counterpart of Girard's translation, to an adjunction between module comprehension categories and comprehension categories. This adjunction corresponds to the translation of MLTT into LMLTT, and it also verifies that our framework refines the categorical semantics of dependent type theories.

To the best of our knowledge, the present work establishes the first complete solution to the long-standing problem of combining linear logic and dependent types in terms of type theory and category theory. One of the main innovations behind this result is that, whilst categorical semantics of MLTT relies on (generalised) cartesian projections in a crucial way, our categorical approach dispenses with them entirely. Accordingly, the standard categorical semantics of dependent product and sum types by adjoints to the indexing functors of projections is no longer valid. Our approach instead interprets linear dependent product and sum types in LMLTT by certain *natural transformations*, where their naturality, instead of the Beck-Chevalley condition, corresponds to the compatibility of the type constructions with substitution. The aforementioned adjoints with the Beck-Chevalley condition have been standard and fundamental in categorical logic for a long time [Jac99], tracing back to Lawvere [Law69, Law70], and hence we believe that our new method in terms of natural transformations will have nontrivial impacts in the field.

1.5 Related work

We have already mentioned the previous approaches to linear dependency based on dual contexts and those based semirings. In the sequel, we list some of the other related work.

Fu et al. [FKS20, FKS22] proposed a linear dependent type theory suitable for quantum circuit programming languages and provided their type theory with operational and denotational semantics. The denotational semantics is given by a class of fibred autonomous (i.e., symmetric monoidal closed) categories over locally cartesian closed categories, where recall that autonomous categories form semantics of the multiplicative fragment of intuitionistic linear logic, and locally cartesian closed categories that of MLTT. Similarly to McBride [McB16], they decorate contexts in the quantum programming language with elements of a semiring, but their types are dependent over the *shape* of the base type (not the base type with the additive unit 0 of the semiring attached). Due to this new feature, it is not straightforward to compare their method with the present work, and we leave it as future work. Nevertheless, there are some apparent differences between the two approaches. For instance, the base of their fibred categorical semantics is locally cartesian closed categories, and their interpretations of linear dependent product and sum types are given in terms of the standard semantics of dependent product and sum types in locally cartesian closed categories. For our aim to refine dependency through linearity, this method is undesirable because it does not decompose dependent product or sum types into more primitive ones. In contrast, our approach defines categorical semantics of linear dependent product and sum types without referring to the semantics of dependent product or sum types, and rather the

former recovers the latter by a generalisation of linear/non-linear adjunction.

Riley in his PhD thesis [Ril22] extended MLTT with linear dependent types with the aim of formalising stable homotopy theory. Instead of elements of a semiring, his approach augments a context with two structures, called a *palette* and *colours* from the palette. Because of these new structures, it is not straightforward to compare his approach with the present work, and we leave it as future work. To date, no semantics of his linear dependent type theory has been given.

Finally, *indexed monoidal categories* have been discovered many times in the literature by various researchers, among which [GG76] is the first one. Because module categories are monoidal categories together with additional structures, indexed module categories are closely related to indexed monoidal categories. For interpreting linear dependent types, however, one usually requires the base category of an indexed monoidal category to be *cartesian* [SHU08] so that *monoidal fibrations* are defined, while the base category of an indexed module categories is monoidal (possibly non-cartesian). Despite this seeming difference between indexed monoidal and module categories, we shall relate the two by inducing indexed monoidal categories with cartesian base categories from indexed module categories through the functor $(\_)^0$ that decorates the base categories with the additive unit 0 of a semiring.

2 Linear Martin-Löf type theory

This section introduces a linear variant of Martin-Löf type theory (MLTT), called ***linear MLTT (LMTLL)***. We assume that the reader is familiar with the basics of MLTT; see the original articles [ML82, ML84b, ML98] by Martin-Löf or the tutorial [Hof97] by Hofmann for the details.

This section proceeds as follows. We first fix the general format or *judgements* of LMLTT in §2.1, and, following this formats, present the rules on judgements for *contexts* in §2.2, *structural rules* in §2.3, and *context morphisms* in §2.4. These rules constitute the core of LMLTT in the sense that they are applicable regardless of postulated types. We next postulate specific type constructions in LMLTT in §2.5, and formulate *commuting conversions* in 2.6. We finally prove some basic properties of LMLTT, e.g., LMLTT recovers MLTT, in §2.7.

For our aim, it is convenient to employ the following categorical formulation of *semirings*:

Definition 2.1 (semirings)**.** A ***semiring*** is a strict symmetric monoidal category $\mathcal{R} = (\mathcal{R}, +, 0)$ with precisely one object $\star_{\mathcal{R}}$ (abbreviated as $\star$) that satisfies the equations

$$(p + q) \times r = (p \times r) + (q \times r) \qquad p \times (q + r) = (p \times q) + (p \times r) \qquad p \times 0 = 0 = 0 \times p,$$

where $\times$ denotes the composition of morphisms, for all morphisms p, q and r in $\mathcal{R}$, and it is said to be ***commutative*** if it additionally satisfies the equation

$$p \times q = q \times p$$

for all morphisms p and q in $\mathcal{R}$. A ***(totally) ordered semiring*** is a semiring $\mathcal{R}$ enriched over the category of (totally) ordered sets and monotone functions.

Remark. In general, strict monoidal categories are rare; *weak* ones abound more. Our semirings are, however, not a counterexample of this phenomenon because each of them has just one object, in which the difference between an equality and an isomorphism between objects vanishes.

Notation. Given a semiring $\mathcal{R} = (\mathcal{R}, +, 0)$, we write $\mathcal{R}_{+,0}$ for the strict symmetric monoidal category $(\mathcal{R}, +, 0)$ or ***additive structure***, and $\mathcal{R}_{\times,1}$ for the strict monoidal category $(\mathcal{R}, \times, 1)$ or ***multiplicative structure***, where $1 := \mathrm{id}_\star$. Abusing notation, we write $r \in \mathcal{R}$ if r is a morphism in $\mathcal{R}$, and pq for $p \times q$. If $\mathcal{R}$ is ordered, then $\leqslant_{\mathcal{R}}$ or $\leqslant$ denotes the order on the hom-set $\mathcal{R}(\star, \star)$.

Example 2.2. If one takes an arbitrary element $\star$ as the unique object, and natural numbers as morphisms, then the set $\mathbb{N}$ of all natural numbers forms an ordered semiring under the standard ordering on $\mathbb{N}$, where addition $+$ and zero 0 constitute the additive structure, and multiplication $\times$ and one 1 the multiplicative structure.

In the remainder of the present section, we fix an arbitrary ordered commutative semiring $\mathcal{R}$.

2.1 Judgements

To explain our ideas, let us first recall that MLTT is a formal system similar to *natural deduction* [Gen35, TS00] except that vertices of a derivation (tree) are ***judgements***, not formulae. Specifically, MLTT consists of the following six judgements (followed by their intended meanings):

1. $\vdash \Gamma\ \mathsf{ctx}$ (Γ is a *context*);
2. $\Gamma \vdash \mathsf{A}\ \mathsf{type}$ (A is a *type* in the context Γ);
3. $\Gamma \vdash \mathsf{a} : \mathsf{A}$ (a is a *term* or *proof* of the type A in the context Γ);
4. $\vdash \Gamma = \Delta\ \mathsf{ctx}$ (Γ and Δ are *(judgmentally) equal* contexts);
5. $\Gamma \vdash \mathsf{A} = \mathsf{A}'\ \mathsf{type}$ (A and A' are *(judgmentally) equal* types in the context Γ);
6. $\Gamma \vdash \mathsf{a} = \mathsf{a}' : \mathsf{A}$ (a and a' are *(judgmentally) equal* terms of the type A in the context Γ),

where judgemental equality is distinguished from *propositional* equality recalled later. A type and a term are primitive concepts, while a context is a derived one: A context is a finite sequence $\mathsf{x}_1 : \mathsf{A}_1, \dots, \mathsf{x}_n : \mathsf{A}_n$ of pairs of a variable x_i and a type A_i, written $\mathsf{x}_i : \mathsf{A}_i$ $(1 \leqslant i \leqslant n)$, such that the variables are pairwise distinct.

Similarly, LMLTT consists of these six judgements except that its contexts, types and terms are decorated with elements of $\mathcal{R}$ for reasoning about *resources*:

1. $\vdash \Gamma\ \mathsf{ctx}$ ($\Gamma = \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_n^{p_n} : \mathsf{A}_n^{p_n}$ is a context);
2. $\Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type}$ (A^p is a type in the context Γ^0);
3. $\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p$ (a^s is a term of the type A^p in the context Γ);
4. $\vdash \Gamma = \Delta\ \mathsf{ctx}$ (Γ and Δ are equal contexts);
5. $\Gamma^0 \vdash \mathsf{A}^p = \mathsf{B}^q\ \mathsf{type}$ (A^p and B^q are equal types in the context Γ^0);
6. $\Gamma \vdash \mathsf{a}_1^{s_1} = \mathsf{a}_2^{s_2} : \mathsf{A}^p$ ($\mathsf{a}_1^{s_1}$ and $\mathsf{a}_2^{s_2}$ are equal terms of the type A^p in the context Γ),

where $p, p_1, \dots, p_n, s, s_1, s_2, q \in \mathcal{R}$, and $\Gamma^q := \mathsf{x}_1^{p_1 q} : \mathsf{A}_1^{p_1 q}, \dots, \mathsf{x}_n^{p_n q} : \mathsf{A}_n^{p_n q}$. Intuitively, an element $p \in \mathcal{R}$ attached to a context, a type or a term signifies the number of times the syntactic object is used. The context Γ^0 of each type A^p is decorated by 0 as in [McB16, Atk18] since variables in Γ are only *parameters* for A^p, not resources, i.e., A^p does not consume any variables in Γ. As we shall verify shortly, if $\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p$, then there is some $\tilde{\Gamma}^q \vdash \tilde{\mathsf{a}}^q : \tilde{\mathsf{A}}^q$ such that $\tilde{\Gamma}^q = \Gamma$, $\tilde{\mathsf{A}}^q = \mathsf{A}^p$ and $\tilde{\Gamma}^q \vdash \mathsf{a}^s = \tilde{\mathsf{a}}^q : \tilde{\mathsf{A}}^q$. We write $\Gamma^0 \vdash \mathsf{A}\ \mathsf{type}$ for $\Gamma^0 \vdash \mathsf{A}^1\ \mathsf{type}$, and $\Gamma \vdash \mathsf{a} : \mathsf{A}^p$ for $\Gamma \vdash \mathsf{a}^1 : \mathsf{A}^p$.

Remark. Strictly speaking, judgements in LMLTT are identified up to the renaming of bound variables to avoid a *variable capture* [B$^+$84] as in the case of MLTT [Hof97, §2]. Accordingly, we implicitly rename bound variables suitably and identify judgements modulo the renaming.

Unlike Atkey [Atk18] $p \in \mathcal{R}$ in a term $\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p$ ranges over *any* element of $\mathcal{R}$, not only 0 or 1. Also, we shall show that unlike McBride [McB16] LMLTT is closed under substitution.

We have not introduced any axioms or rules; we have just taken an overview of the general *formats* of LMLTT. In the rest of this section, we present the rules that constitute LMLTT.

2.2 Contexts

A ***context*** in LMLTT is a finite sequence $\mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \ldots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}$ of pairs of a variable x_i ($i \in \overline{n}$) decorated with an element p_i of $\mathcal{R}$ and a type $\mathsf{A}_\mathsf{i}^{p_i}$, written $\mathsf{x}_\mathsf{i}^{p_i} : \mathsf{A}_\mathsf{i}^{p_i}$, such that the variables are pairwise distinct. We write $(\_)$ for the ***empty context***, i.e., the empty sequence, and we usually omit it if it occurs on the left-hand side of the ***turnstile*** $\vdash$ in a judgement. In this case, we even omit the turnstile $\vdash$ in a judgement as well.

Formally, Figure 1 displays the axioms and the rules on contexts in LMLTT. They are almost

$$(\textsc{Ctx-Emp})\ \frac{}{\vdash (\_)\ \mathsf{ctx}} \qquad (\textsc{Ctx-Ext})\ \frac{\Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type}}{\vdash \Gamma, \mathsf{x}^p : \mathsf{A}^p\ \mathsf{ctx}}\ (\mathsf{x} \text{ does not occur in } \Gamma)$$

$$(\textsc{Ctx-ExtEq})\ \frac{\vdash \Gamma = \Delta\ \mathsf{ctx} \qquad \Gamma^0 \vdash \mathsf{A}^p = \mathsf{B}^q\ \mathsf{type}}{\vdash \Gamma, \mathsf{x}^p : \mathsf{A}^p = \Delta, \mathsf{y}^q : \mathsf{B}^q\ \mathsf{ctx}}\ (\mathsf{x} \text{ does not occur in } \Gamma, \text{ and } \mathsf{y} \text{ does not in } \Delta)$$

Figure 1: The axioms and the rules for contexts in LMLTT

the same as the axioms and the rules on contexts in MLTT; the only difference is the attachment of elements of $\mathcal{R}$. The axiom Ctx-Emp and the rule Ctx-Ext together define that contexts are the finite sequences of variable-type pairs as sketched above. The rule Ctx-ExtEq is a *congruence rule* since it states that the judgmental equality on contexts is preserved under the context extension by Ctx-Ext. Note that we have $\vdash (\_) = (\_)\ \mathsf{ctx}$ by Ctx-Emp and the rule Ctx-EqRefl in §2.3.

Convention. As Hofmann [Hof97] does, we omit the congruence rules for other constructions.

2.3 Structural rules

Next, we collect some standard rules applicable to all types, called ***structural rules***, in Figure 2. The rule Var formulates the reasonable postulate that one may copy an element $\mathsf{x}^p : \mathsf{A}^p$ in the context and paste it to the right-hand side. The remaining part of the context is decorated with 0 as the term only consumes the one $\mathsf{x}^p : \mathsf{A}^p$. In contrast, the corresponding rule in MLTT permits the discard of resources. The other structural rules in LMLTT are the same as the corresponding ones in MLTT except the attachment of elements of $\mathcal{R}$: The next nine rules stipulate that each judgmental equality is an equivalence relation, and the last two rules formalise the assumption that judgements are to be preserved under the exchange of equal contexts and/or types.

2.4 Context morphisms

One of the main differences between MLTT and LMLTT lies in *context morphisms* [Hof97, §2.4]: Those in MLTT are cartesian, while those in LMLTT are monoidal and *non-cartesian* by taking into account the *quantitative* information in *resource-aware* computation.

Let $\Gamma = \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \ldots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}$ and $\Delta = \mathsf{y}_1^{q_1} : \mathsf{B}_1^{q_1}, \ldots, \mathsf{y}_\mathsf{m}^{q_m} : \mathsf{B}_\mathsf{m}^{q_m}$ be contexts. To represent the quantitative information, a context morphism $\phi : \Delta \to \Gamma$ in LMLTT is of the form

$$\mathsf{y}_1^{q_1} : \mathsf{B}_1^{q_1}, \ldots, \mathsf{y}_\mathsf{m}^{q_m} : \mathsf{B}_\mathsf{m}^{q_m} \vdash \mathsf{a}_1^{s_1} : \mathsf{A}_1^{p_1}, \ldots, \mathsf{a}_\mathsf{n}^{s_n} : \mathsf{A}_\mathsf{n}^{p_n}$$

$$(\textsc{Var})\ \frac{\vdash \Gamma, \mathsf{x}^p : \mathsf{A}^p, \Phi\ \mathsf{ctx}}{\Gamma^0, \mathsf{x}^p : \mathsf{A}^p, \Phi^0 \vdash \mathsf{x}^p : \mathsf{A}^p} \qquad (\textsc{Ctx-EqRefl})\ \frac{\vdash \Gamma\ \mathsf{ctx}}{\vdash \Gamma = \Gamma\ \mathsf{ctx}} \qquad (\textsc{Ctx-EqSym})\ \frac{\vdash \Gamma = \Delta\ \mathsf{ctx}}{\vdash \Delta = \Gamma\ \mathsf{ctx}}$$

$$(\textsc{Ctx-EqTrans})\ \frac{\vdash \Gamma = \Delta\ \mathsf{ctx} \qquad \vdash \Delta = \Omega\ \mathsf{ctx}}{\vdash \Gamma = \Omega\ \mathsf{ctx}} \qquad (\textsc{Ty-EqRefl})\ \frac{\Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type}}{\Gamma^0 \vdash \mathsf{A}^p = \mathsf{A}^p\ \mathsf{type}}$$

$$(\textsc{Ty-EqSym})\ \frac{\Gamma^0 \vdash \mathsf{A}^p = \mathsf{B}^q\ \mathsf{type}}{\Gamma^0 \vdash \mathsf{B}^q = \mathsf{A}^p\ \mathsf{type}} \qquad (\textsc{Ty-EqTrans})\ \frac{\Gamma^0 \vdash \mathsf{A}^p = \mathsf{B}^q\ \mathsf{type} \qquad \Gamma^0 \vdash \mathsf{B}^q = \mathsf{C}^r\ \mathsf{type}}{\Gamma^0 \vdash \mathsf{A}^p = \mathsf{C}^r\ \mathsf{type}}$$

$$(\textsc{Tm-EqRefl})\ \frac{\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p}{\Gamma \vdash \mathsf{a}^s = \mathsf{a}^s : \mathsf{A}^p} \qquad (\textsc{Tm-EqSym})\ \frac{\Gamma \vdash \mathsf{a}_1^{s_1} = \mathsf{a}_2^{s_2} : \mathsf{A}^p}{\Gamma \vdash \mathsf{a}_2^{s_2} = \mathsf{a}_1^{s_1} : \mathsf{A}^p}$$

$$(\textsc{Tm-EqTrans})\ \frac{\Gamma \vdash \mathsf{a}_1^{s_1} = \mathsf{a}_2^{s_2} : \mathsf{A}^p \qquad \Gamma \vdash \mathsf{a}_2^{s_2} = \mathsf{a}_3^{s_3} : \mathsf{A}^p}{\Gamma \vdash \mathsf{a}_1^{s_1} = \mathsf{a}_3^{s_3} : \mathsf{A}^p}$$

$$(\textsc{Ty-Conv})\ \frac{\vdash \Gamma = \Delta\ \mathsf{ctx} \qquad \Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type}}{\Delta^0 \vdash \mathsf{A}^p\ \mathsf{type}}$$

$$(\textsc{Tm-Conv})\ \frac{\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p \qquad \vdash \Gamma = \Delta\ \mathsf{ctx} \qquad \Gamma^0 \vdash \mathsf{A}^p = \mathsf{B}^q\ \mathsf{type}}{\Delta \vdash \mathsf{a}^s : \mathsf{B}^q}$$

Figure 2: The structural rules in LMLTT

such that

$$\mathsf{y}_1^0 : \mathsf{B}_1^0, \dots, \mathsf{y}_\mathsf{m}^0 : \mathsf{B}_\mathsf{m}^0, \mathsf{x}_1^0 : \mathsf{A}_1^0, \dots, \mathsf{x}_{\mathsf{i}-1}^0 : \mathsf{A}_{\mathsf{i}-1}^0 \vdash \mathsf{A}_\mathsf{i}^{p_i}\ \mathsf{type}$$

$$\mathsf{y}_1^{q_{1,i}} : \mathsf{B}_1^{q_{1,i}}, \dots, \mathsf{y}_\mathsf{m}^{q_{m,i}} : \mathsf{B}_\mathsf{m}^{q_{m,i}} \vdash \mathsf{a}_\mathsf{i}^{s_i} : \mathsf{A}_\mathsf{i}\{\mathsf{a}_1/\mathsf{x}_1, \dots, \mathsf{a}_{\mathsf{i}-1}/\mathsf{x}_{\mathsf{i}-1}\}^{p_i}$$

$$q_{j,i}, q_j, p_i \in \mathcal{R} \qquad 1 \leqslant i \leqslant n \qquad 1 \leqslant j \leqslant m \qquad \sum_{i=1}^{n} q_{j,i} = q_j.$$

We identify context morphisms up to the componentwise equality between the component terms; this convention corresponds to a congruence rule, so we do not present it formally.

Further, given $r \in \mathcal{R}$, we define another context morphism $\phi^r : \Delta^r \to \Gamma^r$ by

$$\phi^r := \mathsf{y}_1^{q_1 r} : \mathsf{B}_1^{q_1 r}, \dots, \mathsf{y}_\mathsf{m}^{q_m r} : \mathsf{B}_\mathsf{m}^{q_m r} \vdash \mathsf{a}_1^{s_1 r} : \mathsf{A}_1^{p_1 r}, \dots, \mathsf{a}_\mathsf{n}^{s_n r} : \mathsf{A}_\mathsf{n}^{p_n r},$$

which is well-defined as we shall show later.

Example 2.3. Each context $\Gamma = \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}$ has the *identity context morphism*

$$(\mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n} \vdash \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}) : \Gamma \to \Gamma,$$

and together with a term $\Gamma \vdash \mathsf{b}^t : \mathsf{B}^q$ it induces the context morphism

$$(\mathsf{x}_1^{2p_1} : \mathsf{A}_1^{2p_1}, \dots, \mathsf{x}_\mathsf{n}^{2p_n} : \mathsf{A}_\mathsf{n}^{2p_n} \vdash \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}, \mathsf{b}^t : \mathsf{B}^q) : \Gamma^2 \to \Gamma, \mathsf{y}^q : \mathsf{B}^q,$$

where $2 := 1 + 1$.

Given another context $\Delta = \mathsf{y}_1^{q_1} : \mathsf{B}_1^{q_1}, \dots, \mathsf{y}_\mathsf{m}^{q_m} : \mathsf{B}_\mathsf{m}^{q_m}$, there is the *twisting context morphism*

$$(\Gamma, \Delta \vdash \mathsf{y}_1^{q_1} : \mathsf{B}_1^{q_1}, \dots, \mathsf{y}_\mathsf{m}^{q_m} : \mathsf{B}_\mathsf{m}^{q_m}, \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}) : \Gamma, \Delta \to \Delta, \Gamma.$$

Notation. If $\phi : \Delta \to \Gamma$ is a context morphism $\mathsf{y}_1^{q_1} : \mathsf{B}_1^{q_1}, \ldots, \mathsf{y}_m^{q_m} : \mathsf{B}_m^{q_m} \vdash \mathsf{a}_1^{s_1} : \mathsf{A}_1^{p_1}, \ldots, \mathsf{a}_n^{s_n} : \mathsf{A}_n^{p_n}$, and $\Gamma, \Omega \vdash \mathcal{J}$ is a judgement, then the expression

$$\Delta, \Omega\{\phi\} \vdash \mathcal{J}\{\phi\} \tag{4}$$

denotes the judgement obtained from $\Gamma, \Omega \vdash \mathcal{J}$ by (simultaneously) substituting [Hof97, §2.4] a_i for x_i in Ω and in $\mathcal{J}$ for $i = 1, \ldots, n$, which is well-defined as we shall prove shortly.

Another, much more superficial difference between MLTT and LMLTT is that, whilst context morphisms are an auxiliary, derived concept in MLTT, those in LMLTT are *formal objects* with formal rules (similarly to contexts, types and terms) for making the quantitative reasoning of LMLTT explicit. Specifically, LMLTT has the rules displayed in Figure 3 for context morphisms. The axiom CtxMor-Emp yields the empty context morphism. This axiom requires the domain

$$\text{(Mor-Emp)}\ \frac{\vdash \Gamma\ \mathsf{ctx}}{\Gamma^0 \vdash (\_) : (\_)} \qquad \text{(Mor-Ext)}\ \frac{\Delta^d \vdash \phi : \Gamma \qquad \Gamma^0 \vdash \mathsf{B}^q\ \mathsf{type} \qquad \Delta^e \vdash \mathsf{b}^t : \mathsf{B}\{\phi^0\}^q}{\Delta^{d+e} \vdash \phi : \Gamma, \mathsf{b}^t : \mathsf{B}^q}$$

Figure 3: The rules for context morphisms in LMLTT

of the morphism to be of the form Γ^0 because the morphism should not consume any resources. In contrast, MLTT allows any context to be the domain of the empty context morphism, which admits the *discard* of resources. The rule Mor-Ext extends a context morphism by a term in a resource sensitive fashion. By the corresponding rule, MLTT permits the *copying* of resources.

Our context morphisms play a key role for the categorical structure of LMLTT and constitute one of the main differences between LMLTT and the type theory due to McBride or Atkey.

2.5 Type constructions

In the following subsections, we present the axioms and the rules for specific type constructions in LMLTT. As in the case of MLTT, each type construction in LMLTT is defined in terms of ***formation***, ***introduction***, ***elimination*** and ***computation*** rules. The formation rule defines how to construct the type, and the introduction rule stipulates how to generate terms[1] of the type. The elimination and the computation rules describe how to consume the terms and the result of the consumption (in the form of an equation), respectively, both of which are justified by the introduction rule. One may further postulate an optional ***uniqueness*** rule, which imposes some *canonical form* on terms of the type.

Convention. Following a standard convention, henceforth we often omit *evident* judgements (in the sense that they are easily detected from other hypotheses) in the hypotheses of a rule. For instance, instead of presenting both $\Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type}$ and $\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p$, we do only the latter.

Notation. We write $\top$ and 1 for the units of tensor $\otimes$ and with &, respectively, i.e., we swap the traditional notations [Gir87] (similarly to [Tro91, §2.7]), because we find it more systematic.

2.5.1 Top-type

Let us first present the rules on the ***top-type*** $\top$ in Figure 4, As the name indicates, the top-type is a simple type essentially the same as the *multiplicative unit* in linear logic [Gir87]. Alternatively, it is a linear refinement of the *unit-type* in MLTT [Uni13, A.2.8], and therefore intuitively it is a trivially true formula with no information or computation involved.

[1] Strictly speaking, the introduction rule defines *canonical* terms of the type, which in turn defines terms of the type; see [ML84b, NPS90] for the details.

$$(\top\text{-Form})\ \frac{\vdash \Gamma\ \mathsf{ctx}}{\Gamma^0 \vdash \top^p\ \mathsf{type}}\ (p \in \mathcal{R}) \qquad (\top\text{-Intro})\ \frac{\vdash \Gamma\ \mathsf{ctx}}{\Gamma^0 \vdash \star^p : \top^p}\ (p \in \mathcal{R})$$

$$(\top\text{-Elim})\ \frac{\Gamma^g \vdash \mathsf{e}^s : \top^p \qquad \Gamma^h \vdash \mathsf{a}^t : \mathsf{A}^q}{\Gamma^{(g+h)r} \vdash \mathsf{a}^t[\star^p \leftarrow \mathsf{e}^s]^r : \mathsf{A}^{qr}}\ (r \in \mathcal{R})$$

$$(\top\text{-Comp})\ \frac{\Gamma^0 \vdash \star^p : \top^p \qquad \Gamma^h \vdash \mathsf{a}^t : \mathsf{A}^q}{\Gamma^{hr} \vdash \mathsf{a}^t[\star^p \leftarrow \star^p]^r = \mathsf{a}^{tr} : \mathsf{A}^{qr}}\ (r \in \mathcal{R})$$

$$(\top\text{-Uniq})\ \frac{\Gamma^g \vdash \mathsf{e}^s : \top^p \qquad \Gamma^h, \mathsf{x}^p : \top^p \vdash \mathsf{a}^t : \mathsf{A}^q}{\Gamma^{(g+h)r} \vdash \mathsf{a}\{\star/\mathsf{x}\}^t[\star^p \leftarrow \mathsf{e}^s] = \mathsf{a}\{\mathsf{e}/\mathsf{x}\}^t : \top^p}$$

Figure 4: The axioms and the rules for the top-type in LMLTT

The formation rule $\top$-Form constructs the top-type over any context with the zero quantity 0. Note that the top-type is *simple* since it does not contain any variables. The introduction rule $\top$-Intro generates the unique term $\star$ of the top-type, where the context has the zero quantity because $\star$ consumes no resources. The remaining rules are almost the same as the corresponding ones on the unit-type in MLTT except that they take care of the quantitative information.

2.5.2 One-type

Let us next introduce the rules on the ***one-type*** $\mathsf{1}$ in Figure 5. Similarly to the top-type, the one-type is intended to be a trivially true formula. The difference between the two types is that the one-type is essentially the same as the *additive unit* in linear logic, so the introduction rule 1-Intro permits an arbitrary context Γ.

$$(\mathsf{1}\text{-Form})\ \frac{\vdash \Gamma\ \mathsf{ctx}}{\Gamma^0 \vdash \mathsf{1}^p\ \mathsf{type}}\ (p \in \mathcal{R}) \qquad (\mathsf{1}\text{-Intro})\ \frac{\vdash \Gamma\ \mathsf{ctx}}{\Gamma \vdash \bullet^p : \mathsf{1}^p}\ (p \in \mathcal{R}) \qquad (\mathsf{1}\text{-Uniq})\ \frac{\Gamma \vdash \mathsf{o}^s : \mathsf{1}^p}{\Gamma \vdash \mathsf{o}^s = \bullet^p : \mathsf{1}^p}$$

Figure 5: The rules for the one-type in LMLTT

2.5.3 Bottom-type

We next present the rules on the ***bottom-type*** $\bot$ in Figure 6. The intuition is that the bottom-type is a false formula with no information or computation involved. The formation rule $\bot$-Form is just like that of the top- or the one-type, and the elimination rule $\bot$-Elim corresponds to *ex falso*, i.e., anything follows from a contradiction.

$$(\bot\text{-Form})\ \frac{\vdash \Gamma\ \mathsf{ctx}}{\Gamma^0 \vdash \bot^p\ \mathsf{type}}\ (p \in \mathcal{R}) \qquad (\bot\text{-Elim})\ \frac{\Gamma \vdash \mathsf{b}^s : \bot^p \qquad \Delta^0, \mathsf{x}^0 : \bot^0 \vdash \mathsf{A}^q\ \mathsf{type}}{\Delta^0, \Gamma^0, \vdash \mathsf{R}^{\bot}_{\mathsf{A}}(\mathsf{b}^s)^r : \mathsf{A}\{\mathsf{b}/\mathsf{x}\}^{qr}}\ (r \in \mathcal{R})$$

Figure 6: The rules for the bottom-type in LMLTT

2.5.4 Theta-types

Now, let us introduce a generalisation Θ of *tensor* [Gir87] to dependent types, called ***dependent tensor*** or ***theta-types***. LMLTT has the rules on theta-types displayed in Figure 7.

$$(\Theta\text{-Form})\ \frac{\Gamma^0, \mathsf{x}^0 : \mathsf{A}^0 \vdash \mathsf{B}^q\ \mathsf{type}}{\Gamma^0 \vdash (\Theta_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^r\ \mathsf{type}}\ (p, r \in \mathcal{R})$$

$$(\Theta\text{-Intro})\ \frac{\Gamma^0, \mathsf{x}^0 : \mathsf{A}^0 \vdash \mathsf{B}^q\ \mathsf{type} \qquad \Gamma^g \vdash \mathsf{a}^s : \mathsf{A}^p \qquad \Gamma^h \vdash \mathsf{b}^t : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}^q}{\Gamma^{(g+h)r} \vdash (\mathsf{a}^s, \mathsf{b}^t)^r : (\Theta_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^r}\ (r \in \mathcal{R})$$

$$(\Theta\text{-Elim})\ \frac{\Gamma^0, \mathsf{z}^0 : (\Theta_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^0 \vdash \mathsf{C}\ \mathsf{type} \qquad \Gamma^g, \mathsf{x}^{pr} : \mathsf{A}^{pr}, \mathsf{y}^{qr} : \mathsf{B}^{qr} \vdash \mathsf{c}^u : \mathsf{C}\{(\mathsf{x}^p, \mathsf{y}^q)/\mathsf{z}\}^l \qquad \Gamma^h \vdash \mathsf{t}^v : (\Theta_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^r}{\Gamma^{(g+h)k} \vdash \mathsf{c}^u[(\mathsf{x}^p, \mathsf{y}^q)^r \leftarrow \mathsf{t}^v]^k : \mathsf{C}\{\mathsf{t}/\mathsf{z}\}^{lk}}\ (k \in \mathcal{R})$$

$$(\Theta\text{-Comp})\ \frac{\Gamma^0, \mathsf{z}^0 : (\Theta_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^0 \vdash \mathsf{C}\ \mathsf{type} \qquad \Gamma^g, \mathsf{x}^{pr} : \mathsf{A}^{pr}, \mathsf{y}^{qr} : \mathsf{B}^{qr} \vdash \mathsf{c}^u : \mathsf{C}\{(\mathsf{x}^p, \mathsf{y}^q)/\mathsf{z}\}^l \qquad \Gamma^h \vdash (\mathsf{a}^s, \mathsf{b}^t)^r : (\Theta_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^r}{\Gamma^{(g+h)k} \vdash \mathsf{c}^u[(\mathsf{x}^p, \mathsf{y}^q)^r \leftarrow (\mathsf{a}^s, \mathsf{b}^t)^r]^k = \mathsf{c}\{\mathsf{a}/\mathsf{x}\}\{\mathsf{b}/\mathsf{y}\}^{uk} : \mathsf{C}\{(\mathsf{a}^s, \mathsf{b}^t)/\mathsf{z}\}^{lk}}\ (k \in \mathcal{R})$$

$$(\Theta\text{-Uniq})\ \frac{\Gamma^0, \mathsf{z}^0 : (\Theta_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^0 \vdash \mathsf{C}\ \mathsf{type} \qquad \Gamma^g, \mathsf{z}^r : (\Theta_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^r \vdash \mathsf{c}^u : \mathsf{C}^l \qquad \Gamma^h \vdash \mathsf{t}^v : (\Theta_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^r}{\Gamma^{(g+h)k} \vdash \mathsf{c}\{(\mathsf{x}^p, \mathsf{y}^q)/\mathsf{z}\}^u[(\mathsf{x}^p, \mathsf{y}^q)^r \leftarrow \mathsf{t}^v]^k = \mathsf{c}\{\mathsf{t}/\mathsf{z}\}^{uk} : \mathsf{C}\{(\mathsf{a}^s, \mathsf{b}^t)/\mathsf{z}\}^{lk}}\ (k \in \mathcal{R})$$

Figure 7: The rules for theta-type in LMLTT

The intuition is that theta-types represent a dependent conjunction similarly to sigma-types in MLTT. However, being a generalisation of tensor, theta-types are *multiplicative*, while sigma-types are additive. This intuition explains the rules of theta-types.

Notation. We write $\mathsf{A}^p \otimes \mathsf{B}^q$ for a theta-type $\Theta_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q$ if B does not contain the variable x. This convention makes sense because in this case the rules on the theta-type coincides with those on a tensor-type; we leave the details to the reader.

2.5.5 Sigma-types

LMLTT inherits the ***dependent sum*** or ***sigma-types*** Σ form MLTT, which is a dependent type generalisation of *additive conjunction* in linear logic. The rules for sigma-types are displayed in Figure 8. Because contexts in LMLTT are multiplicative and resource sensitive, the elimination rule Σ-Elim and the computation rule Σ-Comp are different from those in MLTT.

2.5.6 Lambda-types

We next introduce ***dependent linear implication*** or ***lambda-types*** Λ, whose rules are displayed in Figure 9. They are a linear refinement of *dependent products* or *pi-types* Π in MLTT.

2.5.7 Exponential-types

As the last type construction in LMLTT, we introduce ***exponential-types*** !. They are essentially the same as *exponential* or *of-course* ! in linear logic, i.e., a countable iteration of tensor $\otimes$. We display the rules on exponential-types in Figure 10.

2.6 Commuting conversions

It is a classic problem in type theory and proof theory that natural deduction systems or λ-calculi do not achieve a unique representation of terms or proofs [GTL89, §10]. From the categorical

$$(\Sigma\text{-Form})\ \frac{\Gamma^0, \mathsf{x}^0 : \mathsf{A}^0 \vdash \mathsf{B}^q\ \mathsf{type}}{\Gamma^0 \vdash (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r\ \mathsf{type}}\ (p, r \in \mathcal{R})$$

$$(\Sigma\text{-Intro})\ \frac{\Gamma^0, \mathsf{x}^0 : \mathsf{A}^0 \vdash \mathsf{B}^q\ \mathsf{type} \qquad \Gamma \vdash \mathsf{a}^s : \mathsf{A}^p \qquad \Gamma \vdash \mathsf{b}^t : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}^q}{\Gamma^r \vdash \langle \mathsf{a}^s, \mathsf{b}^t \rangle^r : (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r}\ (r \in \mathcal{R})$$

$$(\Sigma\text{-Elim})\ \frac{\Gamma \vdash \mathsf{c}^u : (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r}{\Gamma^k \vdash \pi_1(\mathsf{c}^u)^k : \mathsf{A}^{prk}}\ (k \in \mathcal{R}) \qquad (\Sigma\text{-Elim})\ \frac{\Gamma \vdash \mathsf{c}^u : (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r}{\Gamma^k \vdash \pi_2(\mathsf{c}^u)^k : \mathsf{B}\{\pi_1(\mathsf{c}^u)/\mathsf{x}\}^{qrk}}\ (k \in \mathcal{R})$$

$$(\Sigma\text{-Comp})\ \frac{\Gamma^0, \mathsf{x}^0 : \mathsf{A}^0 \vdash \mathsf{B}^q\ \mathsf{type} \qquad \Gamma \vdash \mathsf{a}^s : \mathsf{A}^p \qquad \Gamma \vdash \mathsf{b}^t : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}^q}{\Gamma^r \vdash \pi_1(\langle \mathsf{a}^s, \mathsf{b}^t \rangle)^r = \mathsf{a}^{sr} : \mathsf{A}^{pr}}\ (r \in \mathcal{R})$$

$$(\Sigma\text{-Comp})\ \frac{\Gamma^0, \mathsf{x}^0 : \mathsf{A}^0 \vdash \mathsf{B}^q\ \mathsf{type} \qquad \Gamma \vdash \mathsf{a}^s : \mathsf{A}^p \qquad \Gamma \vdash \mathsf{b}^t : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}^q}{\Gamma^r \vdash \pi_2(\langle \mathsf{a}^s, \mathsf{b}^t \rangle)^r = \mathsf{b}^{tr} : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}^{qr}}\ (r \in \mathcal{R})$$

$$(\Sigma\text{-Uniq})\ \frac{\Gamma \vdash \mathsf{c}^u : (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r}{\Gamma^k \vdash \langle \pi_1(\mathsf{c}^u), \pi_2(\mathsf{c}^u) \rangle^k = \mathsf{c}^{uk} : (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^{rk}}\ (k \in \mathcal{R})$$

Figure 8: The rules for sigma-type in LMLTT

angle, commuting conversions are necessary for syntax to form a term model. For this problem, a standard approach is to introduce *commuting conversions* that identifies terms or proofs module inessential syntactic details. This problem becomes more significant for linear logic, and commuting conversions for a term calculus for intuitionistic linear logic due to Bierman [Bie94, §4.2] are more involved than those of the simply-typed λ-calculus (for intuitionistic logic). Since LMLTT can be seen as a dependent-type generalisation of Bierman's term calculus, it is natural that LMLTT needs commuting conversions to give rise to our categorical structure.

Figure 11 then collects commuting conversions in LMLTT.

2.7 Meta-theoretic properties

This section collects basic properties of LMLTT. We utilise some of them in later sections.

Proposition 2.4 (simultaneous substitution)**.** *For a context morphism $\phi : \Gamma \to \Delta$ and a judgement $\Delta, \Omega \vdash \mathcal{J}$ derivable in LMLTT, the judgement $\Gamma, \Omega\{\phi\} \vdash \mathcal{J}\{\phi\}$ is derivable in LMLTT.*

Proof. By induction on the length of Δ (as in the case of MLTT [Hof97, Proposition 2.12]). □

Corollary 2.5 (weakening and substitution)**.** *The following rules are* admissible *in LMLTT:*

$$(\text{Weak})\ \frac{\Gamma, \Delta \vdash \mathcal{J} \qquad \Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type}}{\Gamma, \mathsf{x}^0 : \mathsf{A}^0, \Delta \vdash \mathcal{J}}\ (\mathsf{x} \notin \Gamma, \Delta)$$

$$(\text{Subst})\ \frac{\Gamma, \mathsf{x}^p : \mathsf{A}^p, \Delta \vdash \mathcal{J} \qquad \Omega \vdash \mathsf{a}^p : \mathsf{A}^p}{\Gamma, \Omega, \Delta\{\mathsf{a}/\mathsf{x}\} \vdash \mathcal{J}\{\mathsf{a}/\mathsf{x}\}}\ (\mathsf{x} \notin \Gamma, \Omega)$$

where $\mathcal{J}$ is the right-hand side of an arbitrary judgement, and $\mathcal{J}\{\mathsf{a}/\mathsf{x}\}$ (respectively, $\Delta\{\mathsf{a}/\mathsf{x}\}$) denotes the (capture-free) substitution *of* a *for* x *in $\mathcal{J}$ (respectively, in Δ) as in [Hof97, §2].*

Proof. By Proposition 2.4. □

$$(\Lambda\text{-Form})\ \frac{\Gamma^0, \mathsf{x}^0 : \mathsf{A}^0 \vdash \mathsf{B}^q\ \mathsf{type}}{\Gamma^0 \vdash (\Lambda_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^r\ \mathsf{type}}\ (p, r \in \mathcal{R}) \qquad (\Lambda\text{-Intro})\ \frac{\Gamma, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^t : \mathsf{B}^q}{\Gamma^r \vdash (\lambda \mathsf{x}^p.\, \mathsf{b}^t)^r : (\Lambda_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^r}\ (r \in \mathcal{R})$$

$$(\Lambda\text{-Elim})\ \frac{\Gamma^g \vdash \mathsf{f}^t : (\Lambda_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^r \qquad \Gamma^h \vdash \mathsf{a}^s : \mathsf{A}^p}{\Gamma^{(g+rh)k} \vdash \mathsf{App}(\mathsf{f}^t, \mathsf{a}^{rs})^k : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}^{qrk}}\ (k \in \mathcal{R})$$

$$(\Lambda\text{-Comp})\ \frac{\Gamma^g, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^t : \mathsf{B}^q \qquad \Gamma^h \vdash \mathsf{a}^s : \mathsf{A}^p}{\Gamma^{(rg+rh)k} \vdash \mathsf{App}((\lambda \mathsf{x}^p.\, \mathsf{b}^t)^r, \mathsf{a}^{rs})^k = \mathsf{b}\{\mathsf{a}/\mathsf{x}\}^{trk} : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}^{qrk}}\ (r, k \in \mathcal{R})$$

$$(\Lambda\text{-Uniq})\ \frac{\Gamma \vdash \mathsf{f}^t : (\Lambda_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^r}{\Gamma^k \vdash (\lambda \mathsf{x}^{pr}.\, \mathsf{App}(\mathsf{f}^t, \mathsf{x}^{pr}))^k = \mathsf{f}^{tk} : (\Lambda_{\mathsf{x}^p:\mathsf{A}^p} \mathsf{B}^q)^{rk}}\ (k \in \mathcal{R})$$

Figure 9: The rules for lambda-type in LMLTT

$$(!\text{-Form})\ \frac{\Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type}}{\Gamma^0 \vdash (!\mathsf{A}^p)^r\ \mathsf{type}}\ (r \in \mathcal{R})$$

$$(!\text{-Intro})\ \frac{\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p}{\Gamma^r \vdash (!\mathsf{a}^s)^r : (!\mathsf{A}^p)^r}\ (r \in \mathcal{R};\ \text{all variables in } \mathsf{a} \text{ are of exponential types})$$

$$(!\text{-Elim})\ \frac{\Gamma \vdash \mathsf{e}^s : (!\mathsf{A}^p)^r \qquad \Delta, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^t : \mathsf{B}^q}{\Delta^{rk}, \Gamma^k \vdash \mathsf{b}^{rt}[(!\mathsf{x}^p)^r \leftarrow \mathsf{e}^s]^k : \mathsf{B}\{\mathsf{e}/\mathsf{x}\}^{rqk}}\ (k \in \mathcal{R})$$

$$(!\text{-Comp})\ \frac{\Gamma \vdash (!\mathsf{a}^s)^r : !\mathsf{A}^{pr} \qquad \Delta, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^t : \mathsf{B}^q}{\Delta^{rk}, \Gamma^k \vdash \mathsf{b}^{rt}[(!\mathsf{x}^p)^r \leftarrow (!\mathsf{a}^s)^r]^k = \mathsf{b}\{\mathsf{a}/\mathsf{x}\}^{rtk} : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}^{rqk}}\ (k \in \mathcal{R})$$

Figure 10: The rules for exponential-type in LMLTT

Proposition 2.6 (multiplication)**.** *Let r be an arbitrary element of the semiring $\mathcal{R}$.*

1. *If $\vdash \Gamma\ \mathsf{ctx}$ is derivable in LMLTT, then so is $\vdash \Gamma^r\ \mathsf{ctx}$;*
2. *If $\Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type}$ is derivable in LMLTT, then so is $\Gamma^0 \vdash \mathsf{A}^{pr}\ \mathsf{type}$;*
3. *If $\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p$ is derivable in LMLTT, then so is $\Gamma^r \vdash \mathsf{a}^{sr} : \mathsf{B}^{pr}$.*

Proof. By induction on derivations in LMLTT. □

Lemma 2.7 (canonical grading)**.** *If $\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p$ is a term, then there is another one $\tilde{\Gamma} \vdash \tilde{\mathsf{a}}^q : \tilde{\mathsf{A}}^q$ such that $\tilde{\Gamma} = \Gamma$, $\tilde{\mathsf{A}}^q = \mathsf{A}^p$ and $\tilde{\Gamma} \vdash \mathsf{a}^s = \tilde{\mathsf{a}}^q : \tilde{\mathsf{A}}^q$. Therefore, if $\phi = (\mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \ldots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n} \vdash \mathsf{b}_1^{t_1} : \mathsf{B}_1^{q_1}, \ldots, \mathsf{b}_\mathsf{m}^{t_m} : \mathsf{B}_\mathsf{m}^{q_m})$ is a context morphism $\Gamma \to \Delta$, then there is another context morphism $\tilde{\phi} = (\mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \ldots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n} \vdash \tilde{\mathsf{b}}_1^{r_1} : \tilde{\mathsf{B}}_1^{r_1}, \ldots, \tilde{\mathsf{b}}_\mathsf{m}^{r_m} : \tilde{\mathsf{B}}_\mathsf{m}^{r_m})$ such that $\phi = \tilde{\phi} : \Gamma \to \Delta$.*

Proof. By induction on derivations of terms in LMLTT. □

Lastly, we define the *composition* of context morphisms as in [Hof97, S 2.4]:

Definition 2.8 (composition of context morphisms)**.** The ***composition*** of context morphisms $\phi : \Gamma \to \Delta$ and $\psi : \Delta \to \Xi$ is the context morphism $\psi \circ \phi : \Gamma \to \Xi$ defined by

$$\psi \circ \phi := \Gamma \vdash \mathsf{c}_1\{\phi\}^{u_1} : \mathsf{C}_1\{\phi\}^{r_1}, \ldots, \mathsf{c}_\mathsf{n}\{\phi\}^{u_n} : \mathsf{C}_\mathsf{n}\{\phi\}^{r_n},$$

$$\mathsf{c}[(\mathsf{x}^p, \mathsf{y}^q)^r \leftarrow \mathsf{t}[(\tilde{\mathsf{x}}^{\tilde{p}}, \tilde{\mathsf{y}}^{\tilde{q}})^{\tilde{r}} \leftarrow \tilde{\mathsf{t}}]] = \mathsf{c}[(\mathsf{x}^p, \mathsf{y}^q)^r \leftarrow \mathsf{t}][(\tilde{\mathsf{x}}^{\tilde{p}}, \tilde{\mathsf{y}}^{\tilde{q}})^{\tilde{r}} \leftarrow \tilde{\mathsf{t}}]$$

$$(\mathsf{a}^s[\star \leftarrow \mathsf{e}^v], \mathsf{b}^t)^u = (\mathsf{a}^s, \mathsf{b}^t)[\star \leftarrow \mathsf{e}^v]^u \qquad (\mathsf{a}^s, \mathsf{b}^t[\star \leftarrow \mathsf{e}^v])^u = (\mathsf{a}^s, \mathsf{b}^t)[\star \leftarrow \mathsf{e}^v]^u$$

Figure 11: Commuting conversions in LMLTT

where $\psi = \Delta \vdash \mathsf{c}_1^{u_1} : \mathsf{C}_1^{r_1}, \ldots, \mathsf{c}_\mathsf{n}^{u_n} : \mathsf{C}_\mathsf{n}^{r_n}$.

Proposition 2.9 (basic properties of composition of context morphisms)**.** *Given context morphisms $\phi, \phi' : \Delta \to \Gamma$, $\psi, \psi' : \Gamma \to \Xi$ and $\varphi : \Xi \to \Omega$ and a judgement $\Xi \vdash \mathcal{J}$, we have*

1. $\psi \circ \phi : \Delta \to \Xi$;
2. $\mathcal{J}\{\psi \circ \phi\} = \mathcal{J}\{\psi\}\{\phi\}$;
3. $\psi \circ \phi = \psi' \circ \phi'$ *if* $\phi = \phi'$ *and* $\psi = \psi'$;
4. $(\varphi \circ \psi) \circ \phi = \varphi \circ (\psi \circ \phi)$;
5. $\mathrm{id}_\Gamma \circ \phi = \phi = \phi \circ \mathrm{id}_\Delta$.

Proof. Straightforward and left to the reader. □

3 Categorical semantics of linear dependency

Having established the syntax of LMLTT, let us proceed to its categorical semantics.

Notation. Given a functor $F : \mathcal{C}^{\mathrm{Op}} \to \mathcal{D}$, we write $\int F$ for the *Grothendieck construction* [Ray71] for F as a contravariant functor, and $\oint F$ for the dual one for F as a covariant functor.

3.1 Module comprehension categories

Our semantics of LMLTT is based on a categorical generalisation of a *module* over a semiring:

Definition 3.1 (module categories)**.** A ***module category*** over a semiring $\mathcal{R}$ consists of

- A symmetric monoidal category $\mathcal{M} = (\mathcal{M}, \otimes, I)$, whose objects are called ***modules***, and morphisms ***linear morphisms***;
- A colax monoidal functor $(\_)^{\cdot} = ((\_)^{\cdot}, \omega, \delta) : \mathcal{R}(\star, \star) \to [\mathcal{M}, \mathcal{M}]$,[2] called the ***grading***, where $[\mathcal{M}, \mathcal{M}]$ is the monoidal category[3] of strong symmetric monoidal endofunctors on $\mathcal{M}$ such that the diagrams in Figure 12 commute for all $\Gamma \in \mathcal{M}$ and $p, q, r \in \mathcal{R}$.

It is said to be ***closed*** if so is the underlying symmetric monoidal category.

Notation. Let $\mathcal{M}$ be a module category over $\mathcal{R}$, and $p \in \mathcal{R}$. We write $\mathcal{M}_p$ for the subcategory of $\mathcal{M}$ whose objects are of the form Γ^p such that $\Gamma \in \mathcal{M}$.

A useful intuition is that a module category over a semiring is a generalisation of the category of vector spaces over a field as well as the category of sets:

[2] Here, the underlying category of the strict monoidal category $\mathcal{R}(\star, \star)$ is discrete.

[3] The monoidal structure of $[\mathcal{M}, \mathcal{M}]$ here comes from that of $\mathcal{M}$. Note that $[\mathcal{M}, \mathcal{M}]$ also has the strict monoidal structure that consists of the composition of functors and the identity functors.

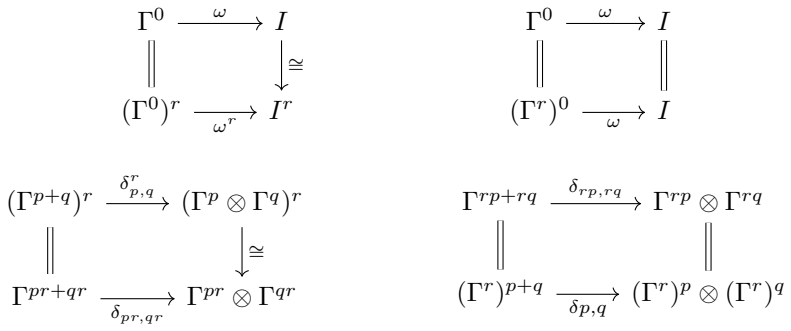


Figure 12: Commutative diagrams for a module category

Example 3.2. The category Vec_K of vector spaces over a field K and linear maps gives rise to a module category over the semiring $\mathbb{N}$:

- The tensor $\otimes$ is the tensor product of vector spaces;
- The unit I is the trivial vector space;
- Given vector spaces $\Gamma, \Delta \in \mathrm{Vec}_K$, a linear function $\phi : \Delta \to \Gamma$ and a natural number $n \in \mathbb{N}$, the vector space $\Gamma^n \in \mathrm{Vec}_K$ is defined by $\Gamma^n := \underbrace{\Gamma \otimes \Gamma \otimes \cdots \otimes \Gamma}_{n}$ $(n > 0)$ and $\Gamma^0 := I$, and the linear function $\phi^n : \Delta^n \to \Gamma^n$ by $\phi^n := \underbrace{\phi \otimes \phi \otimes \cdots \otimes \phi}_{n}$ $(n > 0)$ and $\phi^0 := \mathrm{id}_I$;
- The linear function $\omega : \Gamma^0 = I \to I$ for each $\Gamma \in \mathrm{Vec}_K$ is the identity linear map;
- The linear function $\delta_{p,q} : \Gamma^{p+q} \to \Gamma^p \otimes \Gamma^q$ for each $\Gamma \in \mathrm{Vec}_K$ and $p, q \in \mathbb{N}$ maps

$$v_1 \otimes v_2 \otimes \cdots \otimes v_{p+q} \mapsto (v_1 \otimes v_2 \otimes \cdots \otimes v_p) \otimes (v_{p+1} \otimes v_{p+2} \otimes \cdots \otimes v_{p+q})$$

for all $v_i \in \Gamma$ $(i = 1, 2, \ldots, p+q)$.

Example 3.3. The category Set of sets and functions forms a module category over $\mathbb{N}$:

- The tensor $\times$ is the cartesian product of sets;
- The unit T is an arbitrarily fixed singleton set $\{\star\}$;
- Given sets $X, Y \in \mathrm{Set}$, a function $f : Y \to X$ and a natural number $n \in \mathbb{N}$, the set $X^n \in \mathrm{Set}$ is given by $X^n := \underbrace{X \times X \times \cdots \times X}_{n}$ $(n > 0)$ and $X^0 := T$, and the function $f^n : Y^n \to X^n$ by $f^n := \underbrace{f \times f \times \cdots \times f}_{n}$ $(n > 0)$ and $f^0 := \mathrm{id}_T$;
- The function $\omega : X^0 = T \to T$ for each $X \in \mathrm{Set}$ is the identity map;
- The function $\delta_{p,q} : X^{p+q} \to X^p \times X^q$ for each $X \in \mathrm{Set}$ maps

$$(x_1, x_2, \ldots, x_{p+q}) \mapsto ((x_1, x_2, \ldots, x_p), (x_{p+1}, x_{p+2}, \ldots, x_{p+q}))$$

for all $x_i \in X$ $(i = 1, 2, \ldots, p+q)$.

Next, the following two examples describe links between module categories and LMLTT, and they also explain why the grading of a module category is *colax*:

Example 3.4. LMLTT gives rise to a module category $\mathcal{T}$ over the ordered commutative semiring $\mathbb{N}$ as follows. First, the underlying category $\mathcal{T}$ consists of

- Contexts modulo judgemental equality as objects, where we write $[\Gamma]$ for the equivalence class of each context Γ;
- Context morphisms between contexts Δ and Γ modulo judgemental equality as morphisms $[\Delta] \to [\Gamma]$, where we write $[\phi]$ for the equivalence class of each context morphism ϕ;
- The composition $[\Delta] \overset{[\phi]}{\to} [\Gamma] \overset{[\psi]}{\to} [\Xi]$ of morphisms given by $[\psi] \circ [\phi] := [\psi \circ \phi]$;
- The identity $\mathrm{id}_{[\Gamma]}$ on each object $[\Gamma] = [\mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}]$ is the equivalence class of the context morphism

$$\mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n} \vdash \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}.$$

By Proposition 2.9, these structures constitute a category $\mathcal{T}$. Abusing notation, we omit the bracket $[\_]$ on objects and morphisms. Similarly, by abuse of language, we call an equivalence class of a context (respectively, a context morphism) a context (respectively, a context morphism).

Next, we equip the category $\mathcal{T}$ with the structure of a module category over $\mathbb{N}$:

- The category $\mathcal{T}$ is strict symmetric monoidal, in which the tensor of contexts $\Gamma = \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}$ and $\Delta = \mathsf{y}_1^{q_1} : \mathsf{B}_1^{q_1}, \dots, \mathsf{y}_\mathsf{m}^{q_m} : \mathsf{B}_\mathsf{m}^{q_m}$ is their concatenation $\Gamma, \Delta = \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}, \mathsf{y}_1^{q_1} : \mathsf{B}_1^{q_1}, \dots, \mathsf{y}_\mathsf{m}^{q_m} : \mathsf{B}_\mathsf{m}^{q_m}$, the unit is the empty context $(\_)$, and the symmetry $\Gamma, \Delta \overset{\sim}{\to} \Delta, \Gamma$ is the twisting context morphism

  $$\mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}, \mathsf{y}_1^{q_1} : \mathsf{B}_1^{q_1}, \dots, \mathsf{y}_\mathsf{m}^{q_m} : \mathsf{B}_\mathsf{m}^{q_m} \vdash \mathsf{y}_1^{q_1} : \mathsf{B}_1^{q_1}, \dots, \mathsf{y}_\mathsf{m}^{q_m} : \mathsf{B}_\mathsf{m}^{q_m}, \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \dots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n};$$

- The grading $(\_)^\cdot : \mathbb{N}(\star, \star) \to [\mathcal{T}, \mathcal{T}]$ consists of the maps

  $$r \mapsto (\Gamma \mapsto \Gamma^r) \qquad\qquad \mathrm{id}_r \mapsto (\Delta \overset{\phi}{\to} \Gamma \mapsto \Delta^r \overset{\phi^r}{\to} \Gamma^r),$$

  where $r \in \mathbb{N}$, and the (colax) context morphisms

  $$\omega := (\Gamma^0 \vdash (\_) : (\_)) : \Gamma^0 \to (\_)$$
  $$\delta := (\mathsf{x}_1^{p+q} : \mathsf{A}_1^{p+q}, \dots, \mathsf{x}_\mathsf{n}^{p+q} : \mathsf{A}_\mathsf{n}^{p+q} \vdash \mathsf{x}_1^{p} : \mathsf{A}_1^{p}, \dots, \mathsf{x}_\mathsf{n}^{p} : \mathsf{A}_\mathsf{n}^{p}, \mathsf{x}_1^{q} : \mathsf{A}_1^{q}, \dots, \mathsf{x}_\mathsf{n}^{q} : \mathsf{A}_\mathsf{n}^{p}) : \Gamma^{p+q} \to \Gamma^p, \Gamma^q$$

  given by the rules Mor-Emp and Mor-Ext, respectively. The dual, *lax* ones, $(\_) \to \Gamma^0$ and $\Gamma^p, \Gamma^q \to \Gamma^{p+q}$, in general do not exist. It is easy to see that the four diagrams in Figure 12 commute, and the induced functor $(\_)^r : \mathcal{T} \to \mathcal{T}$ is *strict* symmetric monoidal.

Example 3.5. For each $\Gamma \in \mathcal{T}$, there is the category $\mathrm{Ty}(\Gamma)$ that consists of

- Types $\Gamma^0 \vdash \mathsf{A}^p\ \mathsf{ctx}$ modulo judgemental equality as objects, where we abbreviate the types as A^p, and write $[\mathsf{A}^p]$ for their equivalence classes;
- Terms $\Gamma^u, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^t : \mathsf{B}^q$ for some $u \in \mathcal{R}$ modulo judgemental equality as morphisms $[\mathsf{A}^p] \to [\mathsf{B}^q]$, where we abbreviate them as b^t, and write $[\mathsf{b}^t]$ for their equivalence classes;

- The composition $[\mathsf{A}^p] \xrightarrow{[\mathsf{b}^t]} [\mathsf{B}^q] \xrightarrow{[\mathsf{c}^s]} [\mathsf{C}^r]$ of morphisms $[\Gamma^u, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^t : \mathsf{B}^q]$ and $[\Gamma^v, \mathsf{y}^q : \mathsf{B}^q \vdash \mathsf{c}^s : \mathsf{C}^r]$ defined by

$$[\mathsf{c}^s] \circ [\mathsf{b}^t] := [\Gamma^{u+v}, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{c}\{\mathsf{b}/\mathsf{y}\}^s : \mathsf{C}\{\mathsf{b}/\mathsf{y}\}^r];$$

- The identity $\mathrm{id}_{[\mathsf{A}^p]}$ on each object $[\mathsf{A}^p]$ defined by

$$\mathrm{id}_{[\mathsf{A}^p]} := [\Gamma^0, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{x}^p : \mathsf{A}^p].$$

As in the case of the category $\mathcal{T}$, these structures constitute a well-defined category $\mathrm{Ty}(\Gamma)$. Again, we omit the bracket $[\_]$ and confuse equivalence classes with their representatives.

Remark. One might be tempted to define morphisms $[\mathsf{A}^p] \to [\mathsf{B}^q]$ in $\mathrm{Ty}(\Gamma)$ instead to be terms $\Gamma^0, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^t : \mathsf{B}^q$ modulo judgemental equality. These more restricted morphisms, however, do not capture the quantitative information in the contexts Γ, and in particular this limitation prevents us from capturing lambda-types properly by category theory. On the other hand, one cannot take terms $\Gamma, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^t : \mathsf{B}^q$ modulo judgemental equality as morphisms $[\mathsf{A}^p] \to [\mathsf{B}^q]$ in $\mathrm{Ty}(\Gamma)$ because they are not closed under composition.

This category is symmetric monoidal, where the tensor $\otimes : \mathrm{Ty}(\Gamma) \times \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Gamma)$ maps

$$\big(\mathsf{A}^p, \mathsf{B}^q\big) \mapsto \mathsf{A}^p \otimes \mathsf{B}^q$$
$$\big((\Gamma^0, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{c}^s : \mathsf{C}^l), (\Gamma^0, \mathsf{y}^q : \mathsf{B}^q \vdash \mathsf{d}^t : \mathsf{D}^r)\big) \mapsto \Gamma^0, \mathsf{z} : \mathsf{A}^p \otimes \mathsf{B}^q \vdash (\mathsf{c}^s, \mathsf{d}^t)[(\mathsf{x}^p, \mathsf{y}^q) \leftarrow \mathsf{z}] : \mathsf{C}^l \otimes \mathsf{D}^r,$$

and the unit is (the equivalence class of) the top-type. The tensor preserves composition by a commuting conversion and the computation rule Θ-Comp, and identities by the uniqueness rule Θ-Uniq. We leave it as an exercise to construct the coherence natural isomorphisms for the associativity, the unit law and the symmetry, for which commuting conversions are crucial.

Moreover, this symmetric monoidal category extends to a module category over the semiring $\mathbb{N}$, where the grading functor $(\_)^{\cdot} : \mathbb{N}(\star, \star) \to [\mathrm{Ty}(\Gamma), \mathrm{Ty}(\Gamma)]$ is given by the maps

$$(r, \mathsf{A}^p) \mapsto \mathsf{A}^{pr}$$
$$(\mathrm{id}_r, \Gamma^0, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^q : \mathsf{B}^q) \mapsto \Gamma^0, \mathsf{x}^{pr} : \mathsf{A}^{pr} \vdash \mathsf{b}^{qr} : \mathsf{B}^{qr}$$

together with the evident (colax) morphisms

$$(\mathsf{A}^r)^0 = \mathsf{A}^0 \to \top \qquad\qquad (\mathsf{A}^r)^{p+q} = \mathsf{A}^{pr+qr} \to \mathsf{A}^{pr} \otimes \mathsf{A}^{qr} = (\mathsf{A}^r)^p \otimes (\mathsf{A}^r)^q.$$

given by weakening and the respective rules Θ-Intro and $\top$-Intro. These morphisms make the diagrams in Figure 12 commute; we focus on the third one since the other three are trivial: The composition of $\Gamma^0, \mathsf{z}^r : (\mathsf{A}^p \otimes \mathsf{A}^q)^r \vdash (\mathsf{x}^{pr}, \mathsf{y}^{qr})[(\mathsf{x}^{pr}, \mathsf{y}^{qr}) \leftarrow \mathsf{z}^r] : \mathsf{A}^{pr} \otimes \mathsf{A}^{qr}$ and $\Gamma^0, \mathsf{x}^{(p+q)r} : \mathsf{A}^{(p+q)r} \vdash (\mathsf{x}^p, \mathsf{x}^q)^r : (\mathsf{A}^p \otimes \mathsf{A}^q)^r$ is $\Gamma^0, \mathsf{x}^{(p+q)r} : \mathsf{A}^{(p+q)r} \vdash (\mathsf{x}^{pr}, \mathsf{y}^{qr})[(\mathsf{x}^{pr}, \mathsf{y}^{qr}) \leftarrow (\mathsf{x}^{pr}, \mathsf{y}^{qr})^r] : \mathsf{A}^{pr} \otimes \mathsf{A}^{qr}$, which is equal to $\Gamma^0, \mathsf{x}^{(p+q)r} : \mathsf{A}^{(p+q)r} \vdash (\mathsf{x}^{pr}, \mathsf{y}^{qr}) : \mathsf{A}^{pr} \otimes \mathsf{A}^{qr}$.

We have shown that $\mathrm{Ty}(\Gamma)$ forms a module category over $\mathbb{N}$. Note that, by construction, we have the equation

$$\mathrm{Ty}(\Gamma) = \mathrm{Ty}(\Gamma^0). \tag{5}$$

We next generalise module categories along the path from simple to dependent type theories (or from propositional to predicate logic). To this end, we need a 2-category of module categories:

Definition 3.6 (module functors)**.** A ***module functor*** over a semiring $\mathcal{R}$ is a strong symmetric monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ between module categories over $\mathcal{R}$ together with a family of mediating isomorphisms $[\Gamma]^r_F : F(\Gamma^r) \xrightarrow{\sim} F(\Gamma)^r$ for all $r \in \mathcal{R}$ natural in $\Gamma \in \mathcal{M}$.

Definition 3.7 (module natural transformations)**.** A ***module natural transformation*** over the semiring $\mathcal{R}$ is a symmetric monoidal natural transformation $\alpha : F \Rightarrow G : \mathcal{M} \to \mathcal{M}'$ between module functors over $\mathcal{R}$ such that the diagram

$$\begin{array}{ccc} F(\Gamma^r) & \xrightarrow{[\Gamma]^r_F} & F(\Gamma)^r \\ {\scriptstyle \alpha_{\Gamma^r}}\downarrow & & \downarrow{\scriptstyle \alpha^r_\Gamma} \\ G(\Gamma^r) & \xrightarrow[[\Gamma]^r_G]{} & G(\Gamma)^r \end{array}$$

commutes for all $\Gamma \in \mathcal{M}$ and $r \in \mathcal{R}$.

Proposition 3.8 (2-category of module categories)**.** *(Small) module categories, module functors and module natural transformations over the semiring $\mathcal{R}$ constitute a (large) 2-category* $\mathrm{ModCat}_{\mathcal{R}}$*.*

Proof. First, given small module categories $\mathcal{M} = (\mathcal{M}, \otimes, I)$ and $\mathcal{M}' = (\mathcal{M}', \otimes', I')$ over $\mathcal{R}$, let $\mathrm{ModCat}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}')$ be the category such that

- Objects are module functors $\mathcal{M} \to \mathcal{M}'$;
- Morphisms between objects $F, G : \mathcal{M} \to \mathcal{M}'$ are module natural transformations $F \Rightarrow G$;
- The composition of morphisms $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ is the vertical composition of natural transformations
$$F \overset{\alpha}{\Rightarrow} G \overset{\beta}{\Rightarrow} H : \mathcal{M} \to \mathcal{M}'$$
- The identity on each object $F : \mathcal{M} \to \mathcal{M}'$ is the identity natural transformation $F \Rightarrow F$.

Next, given another small module category $\mathcal{M}'' = (\mathcal{M}'', \otimes'', I'')$ over $\mathcal{R}$, we define the bifunctor $\mathrm{ModCat}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}') \times \mathrm{ModCat}_{\mathcal{R}}(\mathcal{M}', \mathcal{M}'') \to \mathrm{ModCat}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}'')$ that maps

- Each pair of objects $F \in \mathrm{ModCat}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}')$ and $F' \in \mathrm{ModCat}_{\mathcal{R}}(\mathcal{M}', \mathcal{M}'')$ to the composition $F' \circ F$ of strong symmetric monoidal functors
$$\mathcal{M} \xrightarrow{F} \mathcal{M}' \xrightarrow{F'} \mathcal{M}'' \qquad\qquad (F' \circ F)_I := \left(I'' \xrightarrow{F'_{I'}} F'I' \xrightarrow{F'F_I} F'FI \right)$$
$$(F' \circ F)_{A,B} := \left(F'FA \otimes'' F'FB \xrightarrow{F'_{FA,FB}} F'(FA \otimes' FB) \xrightarrow{F'F_{A,B}} F'F(A \otimes B) \right)$$
together with the composition of mediating isomorphisms
$$(F'F\Gamma)^r \xrightarrow{[F\Gamma]^r_{F'}} F'(F\Gamma)^r \xrightarrow{F'[\Gamma]^r_F} F'F(\Gamma^r)$$
for each $\Gamma \in \mathcal{M}$ and $r \in \mathcal{R}$;
- Each pair of morphisms $\alpha : F \to G$ in $\mathrm{ModCat}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}')$ and $\alpha' : F' \to G'$ in $\mathrm{ModCat}_{\mathcal{R}}(\mathcal{M}', \mathcal{M}'')$ to the horizontal composition of natural transformations
$$\alpha' * \alpha : F' \circ F \to G' \circ G.$$

Next, let the functor $\mathrm{id}_{\mathcal{M}} : 1 \to \mathrm{ModCat}_{\mathcal{R}}(\mathcal{M}, \mathcal{M})$ map the unique object of the terminal category 1 to the identity functor $\mathrm{id}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ together with the family of trivial mediating isomorphisms, and the unique morphism in 1 to the identity natural transformation on $\mathrm{id}_{\mathcal{M}}$.

Finally, it is just a routine to verify that these structures constitute a well-defined 2-category as in the case of the 2-category of symmetric monoidal categories. □

We are now ready to generalise module categories and define our basic categorical structure to interpret LMLTT:

Definition 3.9 (module comprehension categories)**.** A ***module pre-comprehension category*** over the semiring $\mathcal{R}$ is a triple $\mathcal{B} = (\mathcal{B}, \mathscr{D}, \_.\_)$ of

- A module category $\mathcal{B} = (\mathcal{B}, \otimes, I, (\_)^{\cdot})$ over $\mathcal{R}$, called the ***base***;
- A functor $\mathscr{D} : \mathcal{B}^{\mathrm{op}} \to \mathrm{ModCat}_{\mathcal{R}}$, called the ***dependency***, such that the diagram

$$\begin{array}{ccccc} \mathcal{B}^{\mathrm{op}} & & \xrightarrow{\quad (\_)^0 \quad} & & \mathcal{B}_0^{\mathrm{op}} \\ {\scriptstyle \mathscr{D}}\downarrow & & & & \downarrow{\scriptstyle \mathscr{D}} \\ \mathrm{ModCat}_{\mathcal{R}} & \xrightarrow[\mathrm{Ob}]{} & \mathrm{Set} & \xleftarrow[\mathrm{Ob}]{} & \mathrm{ModCat}_{\mathcal{R}} \end{array}$$

  in the category Cat commutes, where the functor $\mathrm{Ob} : \mathrm{ModCat}_{\mathcal{R}} \to \mathrm{Set}$ maps each module category $A \in \mathrm{ModCat}_{\mathcal{R}}$ to the set $\mathrm{Ob}(A)$ of all its objects, and each module functor $F : A \to B$ to its object-map $\mathrm{Ob}(f) : \mathrm{Ob}(A) \to \mathrm{Ob}(B)$;

  Notation. We define a functor $\mathscr{D}^+_\Gamma : \mathscr{D}(\Gamma)^{\mathrm{Op}} \to \mathrm{ModCat}_{\mathcal{R}}$ for each $\Gamma \in \mathcal{B}$ by $\mathscr{D}^+_\Gamma := \mathscr{D} \circ (\Gamma.\_)$.
- A functor $\_.\_ : \int_{\mathcal{B}} \mathscr{D}_0 \to \mathcal{B}$, called the ***module pre-comprehension***, where $\int_{\mathcal{B}} \mathscr{D}_0$ denotes the Grothendieck construction for $\mathscr{D}_0 := \mathscr{D} \circ (\_)^0 : \mathcal{B}^{\mathrm{Op}} \to \mathrm{ModCat}_{\mathcal{R}}$,

that satisfies the following axioms:

- (Left unit law) The functor $A \in \mathscr{D}(I) \mapsto I.A \in \mathcal{B}$ defines an equivalence $\mathscr{D}(I) \simeq \mathcal{B}$ in $\mathrm{ModCat}_{\mathcal{R}}$, for which the mapping $\Gamma \mapsto \underline{\Gamma}$ denotes the inverse, and an isomorphism $\Delta \otimes \Gamma \cong \Delta.\underline{\Gamma}\{\omega\}$ natural in $\Gamma, \Delta \in \mathcal{B}$;
- (Right unit law) The unit $\top_\Gamma$ of the module category $\mathscr{D}(\Gamma) = (\mathscr{D}(\Gamma), \otimes_\Gamma, \top_\Gamma, (\_)_{\tilde{\Gamma}})$ for each $\Gamma \in \mathcal{B}$ admits an isomorphism $\Gamma.\top_\Gamma \cong \Gamma$;
- (Associativity) There is a natural transformation $\Theta : \int \mathscr{D}^+ \Rightarrow \mathscr{D} : \mathcal{B}^{\mathrm{Op}} \to \mathrm{Cat}$ that admits an isomorphism $\Gamma.\Theta_\Gamma(A, B) \cong \Gamma.A.B$ natural in $\Gamma \in \mathcal{B}$, $A \in \mathscr{D}(\Gamma)$ and $B \in \mathscr{D}(\Gamma.A)$, where the functor $\int \mathscr{D}^+ : \mathcal{B}^{\mathrm{Op}} \to \mathrm{Cat}$ maps each object $\Gamma \in \mathcal{B}$ to the Grothendieck construction $\int \mathscr{D}^+_\Gamma$, and each morphism $\phi : \Delta \to \Gamma$ in $\mathcal{B}$ to the functor $\int \mathscr{D}^+_\phi := (\_\{\phi\}, \_\{\phi^+\}) : \int \mathscr{D}^+_\Gamma \to \int \mathscr{D}^+_\Delta$ with $\phi^+ := \phi.\mathrm{id} : \Delta.A\{\phi\} \to \Gamma.A$.

The module pre-comprehension is called a ***module comprehension*** if there are bijections

$$\mathscr{D}(I)(\Theta_I(\underline{\Gamma}, A\{\cong\}), A'\{\cong\}) \cong \mathscr{D}(\Gamma)(A, A') \cong \mathscr{D}(\Gamma.A)(\top, A'\{\pi_1^A \circ \cong\}) \tag{6}$$

natural in $\Gamma \in \mathcal{B}$ and $A, A' \in \mathscr{D}(\Gamma)$, where the morphism $\pi_1^A : \Gamma.A^0 \to \Gamma$ in $\mathcal{B}$ is the composition

$$\Gamma.A^0 \overset{\mathrm{id}.\omega}{\to} \Gamma.\top_\Gamma \overset{\sim}{\to} \Gamma.$$

A ***module comprehension category*** over $\mathcal{R}$ is a module pre-comprehension category over $\mathcal{R}$ whose module pre-comprehension is a module comprehension.

Remark. $\mathrm{ModCat}_{\mathcal{R}}$ is not a module category, and the dependency or the module comprehension of a module comprehension category is in general not a module functor. These structures seem impossible to obtain. For instance, there appears no evident choice for the grading on $\mathrm{ModCat}_{\mathcal{R}}$. Also, an obvious choice for the tensor on the result of applying a module pre-comprehension is the componentwise one, but clearly it does not work.

The intuition behind this definition is best provided by the following canonical example:

Example 3.10. If we postulate top- and theta-types, then the module category $\mathcal{T}$ over ω extends to a module comprehension category, which satisfies the left unit law on the nose, as follows:

- The dependency $\mathrm{Ty} : \mathcal{T}^{\mathrm{Op}} \to \mathrm{ModCat}_\omega$ maps each context Γ to the module category $\mathrm{Ty}(\Gamma)$ defined in Example 3.5, and each context morphism $\phi : \Delta \to \Gamma$ to the module functor $\mathrm{Ty}(\phi) : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Delta)$ that maps each type $\Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type}$ to the type $\Delta^0 \vdash \mathsf{A}\{\phi\}^p\ \mathsf{type}$, and each term $\Gamma^0, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^t : \mathsf{B}^q$ to the term $\Delta^0, \mathsf{x}^p : \mathsf{A}\{\phi\}^p \vdash \mathsf{b}\{\phi^u\}^t : \mathsf{B}\{\phi\}^q$. This forms a well-defined functor thanks to Proposition 2.4. Moreover, the dependency clearly satisfies the required commutativity.
- The module comprehension maps each pair (Γ, A^p) of a context Γ and a type $\Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type}$ to the context $\Gamma, \mathsf{x}^p : \mathsf{A}^p$, and a pair (ϕ, a^p) of a context morphism $\phi : \Delta \to \Gamma$ and a term $\Delta^0, \mathsf{y}^q : \mathsf{B}^q \vdash \mathsf{a}^s : \mathsf{A}\{\phi\}^p$ to the context morphism $\phi.\mathsf{a}^s : \Delta, \mathsf{y}^q : \mathsf{B}^q \to \Gamma, \mathsf{x}^p : \mathsf{A}^p$.
- The left unit law is given by the empty context $(\_)$, for which each context $\Gamma = \mathsf{x}_1^{p_1} : \mathsf{A}_1^{p_1}, \ldots, \mathsf{x}_\mathsf{n}^{p_n} : \mathsf{A}_\mathsf{n}^{p_n}$ is mapped to the (simple) type $\vdash \underline{\Gamma}\ \mathsf{type}$ defined inductively by

$$\underline{\epsilon} := \top \qquad\qquad \underline{\Gamma, \mathsf{x}_\mathsf{n}^{p_{n+1}} : \mathsf{A}_{\mathsf{n}+1}^{p_{n+1}}} := \Theta_{\mathsf{x}:\underline{\Gamma}} \mathsf{A}_{\mathsf{n}+1}^{p_{n+1}}\{\iota_\Gamma\},$$

  where $\mathsf{x} : \underline{\Gamma} = (\_).\underline{\Gamma} \xrightarrow{\iota_\Gamma} \Gamma$ is the required isomorphism defined inductively along with $\underline{\Gamma}$.
- The right unit law is satisfied by the top-type.
- The family of all theta-types forms theta.

3.2 Semantic type constructions

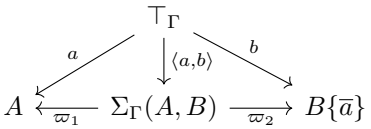
We have seen that the right unit law and the associativity of a module comprehension category correspond in LMLTT to the top-type and theta-types, respectively. In this section, we define categorical semantics of the remaining type constructions.

Convention. We henceforth skip writing evident isomorphisms such as those $\cong$ in (6).

Definition 3.11 (closure)**.** A module comprehension category $\mathcal{B} = (\mathcal{B}, \mathscr{D}, \_.\_)$ is said to be ***closed*** if it has a natural transformation $\Lambda : \int \mathscr{D}^+ \Rightarrow \mathscr{D} : \mathcal{B}^{\mathrm{Op}} \to \mathrm{ModCat}_\mathcal{R}$, called ***lambda***, and a bijection $\mathscr{D}(\Gamma.A)(A'\{\pi_1\}, B) \cong \mathscr{D}(\Gamma)(A', \Lambda(A, B))$ natural in $\Gamma \in \mathcal{B}$, $A, A' \in \mathscr{D}(\Gamma)$ and $B \in \mathscr{D}(\Gamma.A)$.

Example 3.12. Lambda-types form lambda for the module comprehension category $\mathcal{T}$.

Definition 3.13 (sigma)**.** ***Sigma*** for a module comprehension category $\mathcal{B} = (\mathcal{B}, \mathscr{D}, \_.\_)$ refers to a natural transformation $\Sigma : \int \mathscr{D}^+ \Rightarrow \mathscr{D} : \mathcal{B}^{\mathrm{Op}} \to \mathrm{ModCat}_\mathcal{R}$ together with a pair of morphisms $\varpi_1 \in \mathscr{D}(\Gamma)(\Sigma_\Gamma(A, B), A)$ and $\varpi_2 \in \mathscr{D}(\Gamma)(\Sigma_\Gamma(A, B), B\{\overline{\varpi_1}\})$, where $\overline{\varpi_1} := \mathrm{id}.\varpi_1 : \Gamma^0.\Sigma_\Gamma(A, B) \to \Gamma^0.A$, that has a unique morphism $\langle a, b\rangle \in \mathscr{D}(\Gamma)(\top_\Gamma, \Sigma_\Gamma(A, B))$ for a given pair of morphisms $a \in \mathscr{D}(\Gamma)(\top_\Gamma, A)$ and $b \in \mathscr{D}(\Gamma)(\top_\Gamma, B\{\overline{a}\})$ such that the diagram



in $\mathscr{D}(\Gamma)$ commutes.

Example 3.14. Sigma-types form sigma for the module comprehension category $\mathcal{T}$.

Remark. The standard categorical semantics of pi- and sigma-types by adjoints to the indexing functor of the projection morphisms [Jac93] does not work for lambda- and sigma-types because the projection context morphisms $\Gamma.\mathsf{A}^p \to \Gamma$ in general do not exist in LMLTT.

Definition 3.15 (one)**.** ***One*** for a module comprehension category $\mathcal{B} = (\mathcal{B}, \mathscr{D}, \_.\_)$ is a natural transformation $1 : \kappa_T \Rightarrow \mathscr{D} : \mathcal{B}^{\mathrm{Op}} \to \mathrm{ModCat}_{\mathcal{R}}$, where κ_T is the constant functor valued at the terminal category T, such that the hom-set $\mathscr{D}(\Gamma)(\top, 1_\Gamma)$ is singleton for each $\Gamma \in \mathcal{B}$.

Example 3.16. One-types form one for the module comprehension category $\mathcal{T}$.

Definition 3.17 (bottom)**.** ***Bottom*** for a module comprehension $\mathcal{B} = (\mathcal{B}, \mathscr{D}, \_.\_)$ is a natural transformation $\bot : \kappa_T \Rightarrow \mathscr{D} : \mathcal{B}^{\mathrm{Op}} \to \mathrm{ModCat}_{\mathcal{R}}$ such that $\mathscr{D}(\Gamma)(\bot_\Gamma, A) \neq \emptyset$ for all $\Gamma \in \mathcal{B}$ and $A \in \mathscr{D}(\Gamma.\bot_\Gamma)$.

Example 3.18. Bottom-types form bottom for the module comprehension category $\mathcal{T}$.

Definition 3.19 (exponential)**.** ***Exponential*** for a module comprehension $\mathcal{B} = (\mathcal{B}, \mathscr{D}, \_.\_)$ over $\mathcal{R}$ is a limit $! : \mathcal{B} \to \mathcal{B}$ of the grading $(\_)^{\cdot} : \mathcal{R}(\star,\star) \to [\mathcal{B}, \mathcal{B}]$.

Example 3.20. The function that maps types by $\mathsf{A}^p \mapsto !\mathsf{A}^p$ and terms by $\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p \mapsto !\Gamma \vdash !\mathsf{a}^s : !\mathsf{A}^p$ forms exponential for the module comprehension category $\mathcal{T}$.

Theorem 3.21 (completeness of module comprehension categories)**.** *LMLTT forms a module comprehension category $\mathcal{T}$ over the semiring ω, and top-types, theta-types, pi-types, sigma-types, one-types and exponential-types form top, theta, pi, sigma, one and exponential, respectively.*

Theorem 3.22 (dependent linear/non-linear adjunction)**.** *Under construction.*

4 Categorical semantics of linear Martin-Löf type theory

In this last section, we establish categorical semantics of LMLTT in an arbitrary module comprehension category (§4.1) and prove its soundness (§4.2).

4.1 Interpretation

Throughout this section, fix an ordered semiring $\mathcal{R}$ underlying LMLTT and a module comprehension category $\mathcal{B} = (\mathcal{B}, \mathscr{D}, \_.\_)$ over $\mathcal{R}$. Our aim is to define a *semantic map* $[\![\_]\!]_{\mathcal{B}}$ that interprets LMLTT in $\mathcal{B}$. Roughly, the semantic map $[\![\_]\!]_{\mathcal{B}}$ interprets judgements in LMLTT by

$$\begin{aligned}
\vdash \Gamma\ \mathsf{ctx} &\mapsto [\![\Gamma]\!]_{\mathcal{B}} \in \mathcal{B} \\
\Gamma^0 \vdash \mathsf{A}^p\ \mathsf{type} &\mapsto [\![\mathsf{A}]\!]^p_{\mathcal{B}} \in \mathscr{D}([\![\Gamma]\!]_{\mathcal{B}}) \\
\Gamma \vdash \mathsf{a}^s : \mathsf{A}^p &\mapsto [\![\mathsf{a}]\!]^s_{\mathcal{B}} \in \mathscr{D}([\![\Gamma]\!]_{\mathcal{B}})(\top, [\![\mathsf{A}]\!]^p_{\mathcal{B}}) \\
\vdash \Gamma = \Delta\ \mathsf{ctx} &\Rightarrow [\![\Gamma]\!]_{\mathcal{B}} = [\![\Delta]\!]_{\mathcal{B}} \in \mathcal{B} \\
\Gamma^0 \vdash \mathsf{A}^p = \mathsf{B}^q\ \mathsf{type} &\Rightarrow [\![\mathsf{A}]\!]^p_{\mathcal{B}} = [\![\mathsf{B}]\!]^q_{\mathcal{B}} \in \mathscr{D}([\![\Gamma]\!]_{\mathcal{B}}) \\
\Gamma \vdash \mathsf{a}_1^{s_1} = \mathsf{a}_2^{s_2} : \mathsf{A}^p &\Rightarrow [\![\mathsf{a}_1]\!]^{s_1}_{\mathcal{B}} = [\![\mathsf{a}_2]\!]^{s_2}_{\mathcal{B}} \in \mathscr{D}([\![\Gamma]\!]_{\mathcal{B}})(\top, [\![\mathsf{A}]\!]^p_{\mathcal{B}}).
\end{aligned}$$

As in the case of the categorical semantics of MLTT [Hof97, §3.5], a priori we cannot define the semantic map $[\![\_]\!]_{\mathcal{B}}$ for judgements in LMLTT by induction on derivations as a derivation of a judgement may not be unique in the presence of the rules Ty-Con and Tm-Con. Nevertheless, in this section we first define the semantic map $[\![\_]\!]_{\mathcal{B}}$ by induction on derivations, which is not necessarily well-defined, and later in Section 4.2 show that it is well-defined. The latter section also proves the *soundness* of the interpretation.

Definition 4.1 (semantics of contexts)**.** The semantic map $[\![\_]\!]_{\mathcal{B}}$ assigns an object $[\![\Gamma]\!]_{\mathcal{B}} \in \mathcal{B}$ to each context $\vdash \Gamma\ \mathsf{ctx}$ by induction on $|\Gamma|$,

$$[\![(\_)]\!]_{\mathcal{B}} := I \qquad\qquad [\![\Gamma, \mathsf{x}^p : \mathsf{A}^p]\!]_{\mathcal{B}} := [\![\Gamma]\!]_{\mathcal{B}}.[\![\mathsf{A}]\!]^p_{\mathcal{B}}.$$

Definition 4.2 (semantics of context morphisms)**.** The semantic map $[\![\_]\!]_{\mathcal{B}}$ assigns a morphism $[\![\phi]\!]_{\mathcal{B}} : [\![\Delta]\!]_{\mathcal{B}} \to [\![\Gamma]\!]_{\mathcal{B}}$ to each context morphism $\Delta \vdash \phi : \Gamma$ by induction on $|\phi|$:

$$[\![\Delta^0 \vdash (\_) : (\_)]\!]_{\mathcal{B}} := \omega : [\![\Delta]\!]^0_{\mathcal{B}} \to I$$

$$[\![\Delta^{d+e} \vdash \phi : \Gamma, \mathsf{a}^s : \mathsf{A}^p]\!]_{\mathcal{B}} := \big([\![\Delta]\!]^{d+e}_{\mathcal{B}} \xrightarrow{\delta} [\![\Delta]\!]^d_{\mathcal{B}} \otimes [\![\Delta]\!]^e_{\mathcal{B}} \cong [\![\Delta]\!]^d_{\mathcal{B}}.\underline{[\![\Delta]\!]^e_{\mathcal{B}}}\{\omega\} \xrightarrow{\phi.\iota[\![\mathsf{a}]\!]^s_{\mathcal{B}}\{\omega\}} [\![\Gamma]\!]_{\mathcal{B}}.[\![\mathsf{A}]\!]^p_{\mathcal{B}}\big),$$

where $\iota[\![\mathsf{a}]\!]^s_{\mathcal{B}}\{\omega\} \in \mathscr{D}([\![\Delta]\!]^0_{\mathcal{B}})(\underline{[\![\Delta]\!]^e_{\mathcal{B}}}\{\omega\}, [\![\mathsf{A}]\!]^p_{\mathcal{B}}\{[\![\phi]\!]_{\mathcal{B}} \circ \overline{\omega}\})$ is obtained from $[\![\mathsf{a}]\!]^s_{\mathcal{B}} \in \mathscr{D}([\![\Delta]\!]^e_{\mathcal{B}})(\top, [\![\mathsf{A}]\!]^p_{\mathcal{A}}\{[\![\phi]\!]_{\mathcal{B}}\})$ along the natural bijection $\iota : \mathscr{D}([\![\Delta]\!]^e_{\mathcal{B}})(\top, [\![\mathsf{A}]\!]^p_{\mathcal{B}}\{[\![\phi]\!]_{\mathcal{B}}\}) \cong \mathscr{D}(I)(\underline{[\![\Delta]\!]^e_{\mathcal{B}}}, [\![\mathsf{A}]\!]^p_{\mathcal{B}}\{[\![\phi]\!]_{\mathcal{B}}\})$.

Definition 4.3 (semantics of variables)**.** The semantic map $[\![\_]\!]_{\mathcal{B}}$ interprets the rule Var by

$$[\![\Gamma^0, \mathsf{x}^p : \mathsf{A}^p, \Phi^0 \vdash \mathsf{x}^p : \mathsf{A}^p]\!]_{\mathcal{B}} := \pi_2\{\pi_1\} \in \mathscr{D}([\![\Gamma]\!]^0_{\mathcal{B}}.[\![\mathsf{A}]\!]^p_{\mathcal{B}}.[\![\Phi]\!]^0_{\mathcal{B}})(\top, [\![\mathsf{A}]\!]^p_{\mathcal{B}}\{\pi_1\}).$$

Definition 4.4 (semantics of theta-types)**.** The semantic map $[\![\_]\!]$ interprets theta-types by

- (Θ-Form) $[\![\Gamma^0 \vdash (\Theta_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r\ \mathsf{type}]\!]_{\mathcal{B}} := \Theta([\![\Gamma^0 \vdash \mathsf{A}\ \mathsf{type}]\!]^p_{\mathcal{B}}, [\![\Gamma^0, \mathsf{x}^0 : \mathsf{A}^0 \vdash \mathsf{B}\ \mathsf{type}]\!]^q_{\mathcal{B}})^r$;
- (Θ-Intro) $[\![\Gamma^{(g+h)r} \vdash (\mathsf{a}^s, \mathsf{b}^t)^r : (\Theta_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r]\!]_{\mathcal{B}} := ([\![\Gamma^g \vdash \mathsf{a}^s : \mathsf{A}]\!]_{\mathcal{B}} \otimes [\![\Gamma^h \vdash \mathsf{b}^t : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}]\!]_{\mathcal{B}} \circ \delta)^r$;
- (Θ-Elim) $[\![\Gamma^{(g+h)k} \vdash \mathsf{c}^u[(\mathsf{x}^p, \mathsf{y}^q)^r \leftarrow \mathsf{t}^v]^k : \mathsf{C}\{\mathsf{t}/\mathsf{z}\}^{lk}]\!] := \pi_1 \circ [\![\Gamma \vdash \mathsf{c}^u : (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r]\!]^k_{\mathcal{B}}$ (fixed later).

Definition 4.5 (semantics of lambda-types)**.** The semantic map $[\![\_]\!]_{\mathcal{B}}$ interprets the rules on lambda-types by

- (Λ-Form) $[\![\Gamma^0 \vdash (\Lambda_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r\ \mathsf{type}]\!]_{\mathcal{B}} := \Lambda([\![\Gamma^0 \vdash \mathsf{A}\ \mathsf{type}]\!]^p_{\mathcal{B}}, [\![\Gamma^0, \mathsf{x}^0 : \mathsf{A}^0 \vdash \mathsf{B}\ \mathsf{type}]\!]^q_{\mathcal{B}})^r$;
- (Λ-Intro) $[\![\Gamma^r \vdash (\lambda\mathsf{x}^p.\, \mathsf{b}^t)^r : (\Lambda_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r]\!] := \lambda([\![\Gamma, \mathsf{x}^p : \mathsf{A}^p \vdash \mathsf{b}^t : \mathsf{B}^q]\!]_{\mathcal{B}})^r$;
- (Λ-Elim) $[\![\Gamma^{(g+rh)k} \vdash \mathsf{App}(\mathsf{f}^t, \mathsf{a}^{rs})^k : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}^{qrk}]\!]_{\mathcal{B}} := \mathrm{ev} \circ ([\![\Gamma^g \vdash \mathsf{f}^t : (\Lambda_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r]\!]_{\mathcal{B}} \otimes [\![\Gamma^h \vdash \mathsf{a}^s : \mathsf{A}^p]\!]^r_{\mathcal{B}} \circ \delta)^k$.

Definition 4.6 (semantics of sigma-types)**.** The semantic map $[\![\_]\!]$ interprets sigma-types by

- (Σ-Form) $[\![\Gamma^0 \vdash (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r\ \mathsf{type}]\!]_{\mathcal{B}} := \Sigma([\![\Gamma^0 \vdash \mathsf{A}\ \mathsf{type}]\!]^p_{\mathcal{B}}, [\![\Gamma^0, \mathsf{x}^0 : \mathsf{A}^0 \vdash \mathsf{B}\ \mathsf{type}]\!]^q_{\mathcal{B}})^r$;
- (Σ-Intro) $[\![\Gamma^r \vdash \langle \mathsf{a}^s, \mathsf{b}^t \rangle^r : (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r]\!]_{\mathcal{B}} := \langle [\![\Gamma \vdash \mathsf{a}^s : \mathsf{A}]\!]_{\mathcal{B}}, [\![\Gamma \vdash \mathsf{b}^t : \mathsf{B}\{\mathsf{a}/\mathsf{x}\}]\!]^q_{\mathcal{B}} \rangle^r$;
- (Σ-Elim) $[\![\Gamma^k \vdash \pi_1(\mathsf{c}^u)^k : \mathsf{A}^{prk}]\!] := \pi_1 \circ [\![\Gamma \vdash \mathsf{c}^u : (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r]\!]^k_{\mathcal{B}}$ and $[\![\Gamma^k \vdash \pi_2(\mathsf{c}^u)^k : \mathsf{B}\{\pi_1(\mathsf{c}^u)/\mathsf{x}\}^{qrk}]\!] := \pi_2\{[\![\Gamma \vdash \mathsf{c}^u : (\Sigma_{\mathsf{x}^p:\mathsf{A}^p}\mathsf{B}^q)^r]\!]^k_{\mathcal{B}}\}$.

Definition 4.7 (semantics of top-types)**.** The semantic map $[\![\_]\!]_{\mathcal{B}}$ interprets the top-type by

- ($\top$-Form) $[\![\Gamma^0 \vdash 1^p\ \mathsf{type}]\!] := 1^p$
- ($\top$-Intro) $[\![\Gamma \vdash \star^p : 1^p]\!] := \star^p$.
- ($\top$-Elim) $[\![\Gamma \vdash \star^p : 1^p]\!] := \star^p$ (fixed later).

Definition 4.8 (semantics of one-types)**.** The semantic map $[\![\_]\!]_{\mathcal{B}}$ interprets the one-type by

- (1-Form) $[\![\Gamma^0 \vdash 1^p\ \mathsf{type}]\!] := 1^p$
- (1-Intro) $[\![\Gamma \vdash \star^p : 1^p]\!] := \star^p$.

Definition 4.9 (semantics of bottom-types)**.** The semantic map $[\![\_]\!]_{\mathcal{B}}$ interprets the bottom-type by

- ($\bot$-Form) $[\![\Gamma^0 \vdash \bot^p\ \mathsf{type}]\!]_{\mathcal{B}} := \bot^p$;
- ($\bot$-Elim) $[\![\Gamma^r, \Delta^r \vdash \mathsf{R}^{\bot}_{\mathsf{A}}(\mathsf{b}^s)^r : \mathsf{A}\{\mathsf{b}/\mathsf{x}\}^{qr}]\!]_{\mathcal{B}} := \xi \circ [\![\Gamma \vdash \mathsf{b}^s : \bot^p]\!]_{\mathcal{B}}$.

4.2 Soundness

Soundness is proven by the standard method as in the case of MLTT [Hof97].

References

[Atk18] Robert Atkey, *Syntax and semantics of quantitative type theory*, Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, 2018, pp. 56–65.

[AW09] Steve Awodey and Michael A Warren, *Homotopy theoretic models of identity types*, Mathematical proceedings of the cambridge philosophical society, vol. 146, Cambridge University Press, 2009, pp. 45–55.

[B+84] Hendrik Pieter Barendregt et al., *The lambda calculus*, vol. 3, North-Holland, Amsterdam, 1984.

[BBDPH93] Nick Benton, Gavin Bierman, Valeria De Paiva, and Martin Hyland, *A term calculus for intuitionistic linear logic*, International Conference on Typed Lambda Calculi and Applications, Springer, 1993, pp. 75–90.

[BGMZ14] Aloïs Brunel, Marco Gaboardi, Damiano Mazza, and Steve Zdancewic, *A core quantitative coeffect calculus.*, ESOP, vol. 8410, Springer, 2014, pp. 351–370.

[Bie94] Gavin Mark Bierman, *On intuitionistic linear logic*, Tech. report, Citeseer, 1994.

[BP96] Andrew Barber and Gordon Plotkin, *Dual intuitionistic linear logic*, University of Edinburgh, Department of Computer Science, Laboratory for ..., 1996.

[CAB+86] RL Constable, SF Allen, HM Bromley, WR Cleaveland, JF Cremer, RW Harper, DJ Howe, TB Knoblock, NP Mendler, P Panangaden, et al., *Implementing mathematics with the nuprl proof development system.*

[CH88] Thierry Coquand and Gerard Huet, *The calculus of constructions*, Information and computation **76** (1988), no. 2, 95–120.

[Chu40] Alonzo Church, *A formulation of the simple theory of types*, The journal of symbolic logic **5** (1940), no. 2, 56–68.

[CP96] I Cervesato and F Pfenning, *A linear logical framework*, Proceedings 11th Annual IEEE Symposium on Logic in Computer Science, IEEE, 1996, pp. 264–275.

[FKS20] Peng Fu, Kohei Kishida, and Peter Selinger, *Linear dependent type theory for quantum programming languages*, Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science, 2020, pp. 440–453.

[FKS22] ———, *Linear dependent type theory for quantum programming languages*, Logical Methods in Computer Science **18** (2022).

[Gen35] Gerhard Gentzen, *Untersuchungen über das logische schließen. i*, Mathematische zeitschrift **39** (1935), no. 1, 176–210.

[GG76] MF Gouzou and R Grunig, *Fibrations relatives*, Seminaire de Theorie des Categories, dirige par J. Benabou **619** (1976), 620–643.

[Gir87] Jean-Yves Girard, *Linear logic*, Theoretical computer science **50** (1987), no. 1, 1–101.

[GL12] Marco Gaboardi and Ugo Dal Lago, *Linear dependent types and relative completeness*, Logical Methods in Computer Science **8** (2012).

[GS14] Dan R Ghica and Alex I Smith, *Bounded linear types in a resource semiring*, Programming Languages and Systems: 23rd European Symposium on Programming, ESOP 2014, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2014, Grenoble, France, April 5-13, 2014, Proceedings 23, Springer, 2014, pp. 331–350.

[GTL89] Jean-Yves Girard, Paul Taylor, and Yves Lafont, *Proofs and types*, vol. 7, Cambridge university press Cambridge, 1989.

[Her05] Hugo Herbelin, *On the degeneracy of σ-types in presence of computational classical logic*, Typed Lambda Calculi and Applications: 7th International Conference, TLCA 2005, Nara, Japan, April 21-23, 2005. Proceedings 7, Springer, 2005, pp. 209–220.

[Hof97] Martin Hofmann, *Syntax and Semantics of Dependent Types*, Extensional Constructs in Intensional Type Theory, Springer, 1997, pp. 13–54.

[HS98] Martin Hofmann and Thomas Streicher, *The Groupoid Interpretation of Type Theory*, Twenty-five years of constructive type theory (Venice, 1995) **36** (1998), 83–111.

[Jac93] Bart Jacobs, *Comprehension categories and the semantics of type dependency*, Theoretical Computer Science **107** (1993), no. 2, 169–207.

[Jac99] ———, *Categorical Logic and Type Theory*, vol. 141, Elsevier, 1999.

[KL21] Krzysztof Kapulkin and Peter LeFanu Lumsdaine, *The simplicial model of univalent foundations (after voevodsky)*, Journal of the European Mathematical Society **23** (2021), no. 6, 2071–2126.

[KPB15] Neelakantan R Krishnaswami, Pierre Pradic, and Nick Benton, *Integrating linear and dependent types*, ACM SIGPLAN Notices **50** (2015), no. 1, 17–30.

[Law69] F William Lawvere, *Adjointness in foundations*, Dialectica (1969), 281–296.

[Law70] ———, *Quantifiers and sheaves*, Actes du congres international des mathematiciens, Nice, vol. 1, 1970, pp. 329–334.

[Lum09] Peter LeFanu Lumsdaine, *Weak ω-categories from intensional type theory*, Typed lambda calculi and applications, Springer, 2009, pp. 172–187.

[McB16] Conor McBride, *I got plenty o'nuttin'*, A List of Successes That Can Change the World: Essays Dedicated to Philip Wadler on the Occasion of His 60th Birthday (2016), 207–233.

[ML75] Per Martin-Löf, *An Intuitionistic Theory of Types: Predicative Part*, Studies in Logic and the Foundations of Mathematics **80** (1975), 73–118.

[ML82] ———, *Constructive Mathematics and Computer Programming*, Studies in Logic and the Foundations of Mathematics **104** (1982), 153–175.

[ML84a] Per Martin-Lof, *Constructive mathematics and computer programming*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences **312** (1984), no. 1522, 501–518.

[ML84b] Per Martin-Löf, *Intuitionistic Type Theory: Notes by Giovanni Sambin of a series of lectures given in Padova, June 1980*, 1984.

[ML98] ———, *An Intuitionistic Theory of Types*, Twenty-five years of constructive type theory **36** (1998), 127–172.

[ML13] Saunders Mac Lane, *Categories for the working mathematician*, vol. 5, Springer Science & Business Media, 2013.

[NPS90] Bengt Nordström, Kent Petersson, and Jan M Smith, *Programming in martin-löf's type theory, volume 7 of international series of monographs on computer science*, 1990.

[POM14] Tomas Petricek, Dominic Orchard, and Alan Mycroft, *Coeffects: a calculus of context-dependent computation*, ACM SIGPLAN Notices **49** (2014), no. 9, 123–135.

[PS12] KATE PONTO and MICHAEL SHULMAN, *Duality and traces for indexed monoidal categories*, Theory and Applications of Categories **26** (2012), no. 23, 582–659.

[Ray71] Michele Raynaud, *Revêtements étales et groupe fondamental*, Springer-Verlag, 1971.

[Ril22] Mitchell Riley, *A bunched homotopy type theory for synthetic stable homotopy theory*, Ph.D. thesis, Wesleyan University, 2022.

[Sch09] Helmut Schwichtenberg, *Program extraction in constructive analysis*, Springer, 2009.

[SHU08] MICHAEL SHULMAN, *Framed bicategories and monoidal fibrations*, Theory and Applications of Categories **20** (2008), no. 18, 650–738.

[SHU13] ———, *Enriched indexed categories*, Theory and Applications of Categories **28** (2013), no. 21, 616–695.

[SU06] Morten Heine Sørensen and Pawel Urzyczyn, *Lectures on the curry-howard isomorphism*, Elsevier, 2006.

[Tro91] Anne Sjerp Troelstra, *Lectures on linear logic*.

[TS00] Anne Sjerp Troelstra and Helmut Schwichtenberg, *Basic proof theory*, no. 43, Cambridge University Press, 2000.

[Uni13] The Univalent Foundations Program, *Homotopy type theory: Univalent foundations of mathematics*, `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[Vák15] Matthijs Vákár, *A categorical semantics for linear logical frameworks*, Foundations of Software Science and Computation Structures: 18th International Conference, FOSSACS 2015, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2015, London, UK, April 11-18, 2015, Proceedings 18, Springer, 2015, pp. 102–116.

[vdBG11] Benno van den Berg and Richard Garner, *Types are weak ω-groupoids*, Proceedings of the London Mathematical Society **102** (2011), no. 2, 370–394.

[War11] Michael A Warren, *The strict ω-groupoid interpretation of type theory*, Models, logics, and higher-dimensional categories **53** (2011), 291–340.